EMBEDDINGS OF WEIGHTED SOBOLEV SPACES AND GENERALIZED CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

PATRICK J. RABIER

(to appear in J. Analyse Mathématique)

Abstract. We characterize all the real numbers a,b,c and $1\leq p,q,r<\infty$ such that the weighted Sobolev space $W^{(q,p)}_{\{a,b\}}(\mathbb{R}^N\backslash\{0\}):=\{u\in L^1_{loc}(\mathbb{R}^N\backslash\{0\}):|x|^{\frac{a}{q}}u\in L^q(\mathbb{R}^N),|x|^{\frac{b}{p}}\nabla u\in(L^p(\mathbb{R}^N))^N\}$ is continuously embedded into $L^r(\mathbb{R}^N;|x|^c dx):=\{u\in L^1_{loc}(\mathbb{R}^N\backslash\{0\}):|x|^{\frac{c}{r}}u\in L^r(\mathbb{R}^N)\}$, with norm $||\cdot||_{c,r}$.

Except when $N\geq 2$ and $a=c=b-p=-N$, it turns out that this embedding is equivalent to the multiplicative inequality $||u||_{c,r}\leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$ for some suitable $\theta\in[0,1]$, often but not always unique. If $a,b,c>-N$, then $C^\infty_0(\mathbb{R}^N)\subset W^{(q,p)}_{\{a,b\}}(\mathbb{R}^N\backslash\{0\})\cap L^r(\mathbb{R}^N;|x|^c dx)$ and such inequalities for $u\in C^\infty_0(\mathbb{R}^N)$ are the well-known Caffarelli-Kohn-Nirenberg inequalities, but their generalization to $W^{(q,p)}_{\{a,b\}}(\mathbb{R}^N\backslash\{0\})$ cannot be proved by a denseness argument. Without assuming $a,b,c>-N$, the inequalities are essentially new even when $u\in C^\infty_0(\mathbb{R}^N\backslash\{0\})$, although a few special cases are known, most notably the Hardy-type inequalities when $p=q$.

In a different direction, the embedding theorem easily yields a generalization when the weights $|x|^a,|x|^b$ and $|x|^c$ are replaced by more general weights w_a,w_b and w_c, respectively, having multiple power-like singularities at finite distance and at infinity.

1. Introduction

If $d\in\mathbb{R}$ and $1\leq s<\infty$, let $||\cdot||_{d,s}$ denote the norm of the space $L^s(\mathbb{R}^N;|x|^d dx)$, where the $|x|^d dx$ -measure of $\{0\}$ is defined to be 0 (which must be specified if $d\leq -N$). With this definition, $u\in L^s(\mathbb{R}^N;|x|^d dx)$ if and only if $|x|^{\frac{d}{s}}u\in L^s(\mathbb{R}^N)$ and $||u||_{d,s}=||\,|x|^{\frac{d}{s}}u||_s$, where $||\cdot||_s:=||\cdot||_{0,s}$. Throughout the paper, $\mathbb{R}^N_*:=\mathbb{R}^N\backslash\{0\}$.

Given $a,b\in\mathbb{R}$ and $1\leq p,q<\infty$, consider the weighted Sobolev space

$$W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*):= \{u\in L^1_{loc}(\mathbb{R}^N_*):u\in L^q(\mathbb{R}^N;|x|^a dx),\quad \nabla u\in(L^p(\mathbb{R}^N;|x|^b dx))^N\},\tag{1.1}$$

equipped with the norm

$$||\,u||_{a,q}+||\,\nabla u||_{b,p}.\tag{1.2}$$

1991 *Mathematics Subject Classification.* 46E35.

Key words and phrases. Weighted Sobolev space, embedding, Hardy-type inequalities, CKN inequalities.

I am grateful to Prof. V. Maz'ya for valuable information about the early historical development of the subject and other connections with the existing literature.

Since $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ may contain functions which are not locally integrable near 0 and hence not distributions on $\mathbb{R}^N$, it is generally larger than the space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N)$ (self-explanatory notation) which, incidentally, is not always complete.

In this paper, we characterize all the real numbers a, b, c and $1 \leq p, q, r < \infty$ such that

$$W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx),$$

where "$\hookrightarrow$" denotes continuous embedding. This provides sufficient conditions for $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, but their necessity is not investigated.

In spite of the large literature devoted to embeddings of weighted Sobolev spaces, there seems to be little that addresses and resolves the exact same question in special cases. While most results allow for weights satisfying general properties, they also incorporate a number of restrictive hypotheses which are rarely necessary. Only a few are applicable to the whole -or punctured- space and even fewer accommodate weights which, like all nontrivial power weights, exhibit singularities at 0 and infinity simultaneously. This is especially true when more than one weight (here, $a \neq b$) or more than one order of integration (i.e., $p \neq q$) is involved in the source space. In addition, the weighted spaces are often defined to be the unknown closure of some subspace of smooth (enough) functions, as indeed the denseness issue is a notorious difficulty ([30]). In particular, this is the definition chosen in [17] (see also the more recent and expanded book [18]), except in the unweighted case.

Before continuing this discussion, we shall state the embedding theorem. In addition to the standard notation

$$p^* = \infty \text{ if } p \geq N \text{ and } p^* = \frac{Np}{N-p} \text{ if } 1 \leq p < N,$$

we denote by c^0 and c^1 the two points

$$c^0 := \frac{r(a+N)}{q} - N \quad \text{and} \quad c^1 := \frac{r(b-p+N)}{p} - N, \tag{1.3}$$

where it is understood that a, b, p, q and r are given. The points c^0 and c^1 are distinct if and only if $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. If so and if c is in the closed interval with endpoints c^0 and c^1, we set

$$\theta_c := \frac{c - c^0}{c^1 - c^0}, \tag{1.4}$$

so that $\theta_c \in [0, 1]$ and that

$$c = \theta_c c^1 + (1 - \theta_c) c^0. \tag{1.5}$$

Observe that $\theta_{c^0} = 0$ and $\theta_{c^1} = 1$ and that, by (1.3), (1.4) and (1.5),

$$\frac{c+N}{r} = \theta_c \frac{b-p+N}{p} + (1 - \theta_c) \frac{a+N}{q}. \tag{1.6}$$

Theorem 1.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be given ($1 \leq p < \infty$ and $0 < q, r < \infty$ if $N = 1$). Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) if and only if $r \leq \max\{p^*, q\}$ and one of the following conditions[1] holds:*
(i) a and $b - p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is

[1]The overlap between conditions (iii), (iv) and (v) makes for a simpler and clearer statement.

in the open interval with endpoints c^0 and c^1 and $\theta_c\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \leq \frac{1}{r}-\frac{1}{q}$.
(ii) a and $b-p$ are strictly on opposite sides of $-N$ (hence $\frac{a+N}{q} \neq \frac{b-p+N}{p}$), c is in the open interval with endpoints c^0 and $-N$ and $\theta_c\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \leq \frac{1}{r}-\frac{1}{q}$.
(iii) $r = q$ and $c = a$ $(= c^0)$.
(iv) $p \leq r \leq p^*, a \leq -N$ and $b-p < -N$ or $a \geq -N$ and $b-p > -N, c = c^1$.
(v) $(\max\{p^*, q\} \geq)$ $r \geq \min\{p, q\}, \frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$ and $c = c^1$ $(= c^0)$.
(vi) $a = -N, b = p-N, q < r \leq p^*$ and $c = c^1$ $(= c^0 = -N)$.

Since r is finite, $r = p^*$ is impossible when $p \geq N$. The set of admissible values of c is an interval (possibly $\emptyset$, see Remark 1.1), of which c^0, c^1 and $-N$ may or may not be endpoints, but never interior points. When c^0 or c^1 are endpoints, their admissibility is decided by parts (iii) to (vi). Endpoints other than c^0, c^1 or $-N$ are always admissible, but $-N$ is never admissible when $a \neq -N$. If $a = -N$, then $-N$ is admissible only in the trivial case (iii) and the exceptional case (vi).

Apparently, aside from the trivial part (iii), only parts (v) and (vi) of Theorem 1.1 when $q = p$ (hence $a = b-p$) are known with nontrivial weights. See Opic and Kufner [22, p. 291], where the result is credited to Opic and Gurka [21]. Curiously, if $b-p \neq -N$ and $a_q := \frac{q(b-p+N)}{p} - N$, part (v) shows that the space $W^{1,(q,p)}_{\{a_q,b\}}(\mathbb{R}^N_*)$ is independent of $q \in [p, p^*], q < \infty$, with equivalent norms as q is varied. When $N = 1$, part (iv) can -and will- be deduced from an inequality of Bradley [5]. Related, but different, work is discussed further below.

In the unweighted case $a = b = c = 0$ and if $p = q$ and $N \geq 2$ (a minor point), Theorem 1.1 gives again $W^{1,p}(\mathbb{R}^N_*) = W^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ if and only if ($r < \infty$ and) $p \leq r \leq p^*$ (Subsection 11.1). If $p \neq q$ (and still $a = b = c = 0$), Theorem 1.1 is akin to embedding theorems in [2], [3].

Remark 1.1. *If $r \leq \min\{p^*, q\}$, then $\theta_c\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \leq \frac{1}{r}-\frac{1}{q}$ for every c between c^0 and c^1. In contrast, all the conditions of Theorem 1.1 fail (i.e., no embedding holds for any c) if $p < N$ and $r > \max\{p^*, q\}$ or if either (i) $p < N, r = p^* > q, b-p = -N \neq a$ or (ii) $q < r \leq p^*, a$ and $b-p$ are strictly on opposite sides of $-N$ (hence θ_{-N} is defined) and $\theta_{-N}\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \geq \frac{1}{r}-\frac{1}{q}$.*

When $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, a simple rescaling shows (Corollary 2.2) that the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ is equivalent to the multiplicative inequality

$$||u||_{c,r} \leq C||\nabla u||^{\theta_c}_{b,p}||u||^{1-\theta_c}_{a,q}, \tag{1.7}$$

rather than just $||u||_{c,r} \leq C\left(||u||_{a,q} + ||\nabla u||_{b,p}\right)$.

When $a, b, c > -N$ and $u \in C^\infty_0(\mathbb{R}^N)$, (1.7) is one of the well-known Caffarelli-Kohn-Nirenberg (CKN for short) inequalities in [6]. Therefore, parts (i) and (ii) of Theorem 1.1 give necessary and sufficient conditions for the validity of the CKN inequality (1.7) when $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, but without the restriction $a, b, c > -N$ and for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Note that $C^\infty_0(\mathbb{R}^N) \subset W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ when $a, b > -N$, so that even in this case, (1.7) is a genuine generalization. As already pointed out, it does not follow by a denseness argument without many extra conditions ($\mathbb{R}^N_*$ replaced by $\mathbb{R}^N, p = q$, $a = b$ and $|x|^a$ an A_p weight, i.e. $-N < a < (p-1)N$; see [11, Theorem 1.27] or [20]). The denseness of $C^\infty_0(\mathbb{R}^N)$ is obviously meaningless when

$a \leq -N$ or $b \leq -N$ while that of $C_0^\infty(\mathbb{R}_*^N)$, always contained in $W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N)$, is generally false (see Subsection 11.3) and hence definitely not a viable approach.

Inequalities of CKN type have been discussed earlier, beginning with the 1961 work of Il'in [12, Theorem 1.4], who proved (with c^1 given by (1.3)) $||u||_{c^1,r,G} \leq C||\nabla u||_{b,p,\Omega}$ when Ω is a fairly general open subset of $\mathbb{R}^N$, G is a bounded measurable subset of Ω and u is C^1. There are further limitations about b, p and r, but the result has various generalizations when higher order derivatives are involved, or when G is a bounded subset of a section of Ω by a lower-dimensional hyperplane. Results of a somewhat similar nature are proved in [17, Section 2.1.6], [18] when $\Omega = \mathbb{R}^N$ and $u \in C_0^\infty(\mathbb{R}^N)$.

When Ω is an open subset of $\mathbb{R}^N$, μ and ν are nonnegative Borel measures, $\Phi \geq 0$ is continuous and positively homogeneous of degree 1 in its second argument and $\frac{1}{r} \leq \frac{\theta}{p} + \frac{1-\theta}{q}$, Maz'ya [16, Theorem 9] (reproduced in [17, p.127] and [18]) gives interesting *necessary and sufficient* conditions for the inequality

$$||u||_{L^r(\Omega;\mu)} \leq C \left(\int_\Omega \Phi(x, \nabla u)^p dx \right)^{\frac{\theta}{p}} ||u||_{L^q(\Omega;\nu)}^{1-\theta}, \tag{1.8}$$

to hold for $u \in C_0^\infty(\Omega)$. When $\Omega = \mathbb{R}_*^N, \mu(E) = \int_E |x|^c dx, \nu(E) = \int_E |x|^a dx$ and $\Phi(x, y) = |x|^{\frac{b}{p}}|y|$, the setting of Theorem 1.1 is recovered.

Maz'ya's conditions for (1.8) are expressed in terms of the (p, Φ)-capacity of "admissible" sets and their μ and ν measures. As early as 1960, he noted in [15] that such conditions could be used to prove the equivalence between various inequalities (e.g., Sobolev and Nash). This kind of equivalence has since been revisited by a number of authors. For example, when $a = c$, it follows from Bakry *et al.* [1] that if the inequality $||u||_{a,r} \leq C||\nabla u||_{b,p}^\theta ||u||_{a,q}^{1-\theta}$ holds when $q = q_0, r = r_0, \theta = \theta_0$ and (say) u is a Lipschitz continuous function with compact support, then the same inequality continues to hold for a family of other values of q, r and θ. Once again, denseness issues are an obstacle to extending this property to the spaces $W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N)$ unless $a = b = c = 0$ (unweighted case).

The connection of this work with the CKN inequalities can be found in some of the preliminary results in [6] which, possibly in generalized form, are also useful for the proof of Theorem 1.1. However, without the compactness of the supports and other key assumptions, a mere tweaking of the arguments of [6] is not possible.

In the next section, we show that (1.7) is equivalent to an embedding inequality and that the hypotheses of Theorem 1.1 are necessary. The necessity of $r \leq \max\{p^*, q\}$ and of $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$ in parts (i) and (ii) of Theorem 1.1 follows very simply from (1.7) and a remark in [6] used here in a more general framework (Theorem 2.3 (i)). A variant of it proves the necessity of $r \leq \max\{p^*, q\}$ in the remaining cases (Theorem 2.3 (ii)).

The verification of the sufficiency is demanding. The general idea is first to prove Theorem 1.1 for radially symmetric functions. Once this is done, there are two different ways to proceed. The first one is to reduce the problem to the symmetric case by a *suitable* radial symmetrization. This works when $1 \leq r \leq \min\{p, q\}$. The second option is to prove an independent embedding theorem for a direct complement of the subspace of radially symmetric functions. This can be done, based on ideas in [6], under assumptions about p, q and r that rule out $r < \min\{p, q\}$. This is why it is crucial that this case can be settled by other arguments.

The proof of the embedding theorem for radially symmetric functions and, next, by radial symmetrization, requires some preliminaries. It is more natural to work with the larger spaces (the domain $\mathbb{R}_*^N$ is not mentioned for simplicity)

$$\widetilde{W}^{1,(q,p)}_{\{a,b\}} := \{u \in L^1_{loc}(\mathbb{R}^N_*) : u \in L^q(\mathbb{R}^N; |x|^a dx), \quad \partial_\rho u \in L^p(\mathbb{R}^N; |x|^b dx)\}, \tag{1.9}$$

equipped with the norm

$$||u||_{\{a,b\},(q,p)} := ||u||_{a,q} + ||\,\partial_\rho u||_{b,p}, \tag{1.10}$$

where $\partial_\rho u := \nabla u \cdot \frac{x}{|x|}$ is the radial derivative of u. Since $|x|^{-1}x$ is a smooth field on $\mathbb{R}^N_*$, this definition makes sense for every distribution u on $\mathbb{R}^N_*$.

When $0 < q < 1$, the definitions (1.1) and (1.9) can still be used, but (1.2) and (1.10) are only quasi-norms. The equivalence between continuity and boundedness for linear operators remains true in quasi-normed spaces. For more details about such spaces, see [4] or [24].

The spaces $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ and $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ contain the same radially symmetric functions and the induced (quasi) norms are the same, because $\nabla u = (\partial_\rho u)\frac{x}{|x|}$ when u is radially symmetric. Thus, when referring to radially symmetric functions, the ambient space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ or $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is unimportant.

In the next section, the basic features of a related space $\widetilde{W}^{1,p}_{loc}(\mathbb{R}^N_*)$ (abbreviated $\widetilde{W}^{1,p}_{loc}$) are discussed, along with some of their implications regarding $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$. This material is directly relevant to the proof of the main results of Sections 4 and 5.

Necessary and sufficient conditions for the continuity of the embedding of the subspace of radially symmetric functions when $q, r > 0$ and $p \geq 1$ are given in Theorem 4.7. Of course, this is a (barely) disguised form of Theorem 1.1 when $N = 1$. Compared with the treatment of the same problem in [6], convenient tools (e.g., radial integration by parts) cannot be used and some estimates (e.g., of $|u(0)|$) make no longer sense. For that reason, our approach is technically completely different.

The proof of Theorem 1.1 for arbitrary N begins in Section 5, where the case $1 \leq r \leq \min\{p, q\}$ is considered. As mentioned before, this is done by radial symmetrization, though not in the obvious way (Lemma 5.1). The result (Theorem 5.2) is more general and sharper than the corresponding part of Theorem 1.1 since it establishes the continuous embedding of the larger space $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$, with a weaker norm, into $L^r(\mathbb{R}^N; |x|^c dx)$ under the conditions already necessary for the embedding of $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Thus, the embedding is obtained without assuming the integrability of the first derivatives, except for just the radial one.

The case when $r > \min\{p, q\}$ is split into the three parts: $p < r \leq q$ (Theorem 7.1), $r > q$ and $r \geq p$ (Theorem 8.3) and $q < r < p$ (Theorem 9.1). If $p = q$, Sections 7 and 9 can be skipped with no prejudice. A preliminary embedding lemma for functions with null radial symmetrization, essentially due to Caffarelli, Kohn and Nirenberg, is proved in Section 6 (Lemma 6.1), then rephrased in a more convenient way (Corollary 6.2). The technical steps are simple, but cannot be repeated with the larger space $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$. The proofs of Theorem 7.1 (when $p < r \leq q$) and Theorem 9.1 (when $1 \leq q < p < r$) also heavily rely on Theorem 5.2 (when $1 \leq r \leq \min\{p, q\}$, but with other parameters).

The relationship between Theorem 1.1 and the CKN inequalities does not stop with (1.7) when $\frac{a+N}{q} \neq \frac{b-p+N}{p}$: In Section 10, we show that the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N;|x|^c dx)$ continues to be equivalent to a multiplicative inequality $||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$ for some suitable $\theta \in [0,1]$ when $\frac{a+N}{q} = \frac{b-p+N}{p}$ (Theorem 10.2), except when $N \geq 2$ and $a = c = b - p = -N$ (Theorem 10.3). Of course, θ is no longer θ_c in (1.4), which is not defined, and it may not always be unique (Remark 10.1) When $\theta = 1$, this is an N-dimensional weighted Hardy inequality more general than those in the current literature ([9], [22]). The case when $u \in C^\infty_0(\mathbb{R}^N_*), p = q = r = 2, c = \frac{a+b}{2} - 1$ and $\theta = \frac{1}{2}$ was recently investigated by Catrina and Costa [7].

In Section 11, three special cases are discussed and the (simple) generalization when $|x|^a, |x|^b$ and $|x|^c$ are replaced by weights w_a, w_b and w_c having multiple power-like singularities is briefly sketched.

1.1. **Notation.** Throughout the paper, $C > 0$ denotes a constant which, as is customary, may have different values in different places. If $k \geq 1$ is a real number, $k' \leq \infty$ will always denote the Hölder conjugate of k. Also, $\zeta \in C^\infty_0(\mathbb{R}^N)$ is chosen once and for all such that $0 \leq \zeta \leq 1$ is radially symmetric, $\zeta(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\zeta(x) = 0$ if $|x| \geq 1$. Naturally, we shall also use the notation introduced more formally earlier on. Up to and including Section 4, we shall frequently refer to the Kelvin transform, defined in the following remark.

Remark 1.2. *The Kelvin transform $x \mapsto |x|^{-2}x$ on $\mathbb{R}^N_*$ is an isometry from $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ onto $\widetilde{W}^{1,(q,p)}_{\{-2N-a,2p-2N-b\}}$ and from $L^r(\mathbb{R}^N;|x|^c dx)$ onto $L^r(\mathbb{R}^N;|x|^{-2N-c}dx)$ for all values of the parameters. As a result, in many proofs that split into two complementary cases, it will be enough to discuss only one of them, because the other follows from this isometry.*

2. Necessary conditions for continuous embedding

In this section, we prove that the conditions given in Theorem 1.1 are necessary.

Theorem 2.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be given. Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ (hence a fortiori $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$) is not contained $L^r(\mathbb{R}^N;|x|^c dx)$ if:*
(i) c does not belong to the closed interval with endpoints c^0 and c^1.
(ii) $b - p \leq -N < a$ or $b - p \geq -N > a$ and c does not belong to the interval with endpoints c^0 (included) and $-N$ (not included).
Furthermore, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ (hence a fortiori $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$) is not continuously[2] embedded into $L^r(\mathbb{R}^N;|x|^c dx)$ if:
(iii) $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, $c = c^0$ and $r \neq q$ (if $r = q$, then $c^0 = a$ and the embedding is trivial).
(iv) $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, $c = c^1$ and $r < p$.
(v) $\frac{a+N}{q} = \frac{b-p+N}{p}, r < \min\{p,q\}$ and $c = c^0$ $(= c^1)$.
(vi) $a = -N, b = p - N, r < q$ and $c = c^0$ $(= c^1 = -N)$.

Proof. (i) If $c < \min\left\{c^0, c^1\right\}$, let $u(x) := |x|^{-\frac{c+N}{r}}\zeta(x)$ with ζ as in subsection 1.1. Then, $u \notin L^r(\mathbb{R}^N;|x|^c dx)$ since $|x|^c|u(x)|^r = |x|^{-N}$ on a neighborhood of 0,

[2] In principle at least, that does not rule out $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \subset L^r(\mathbb{R}^N;|x|^c dx)$.

but $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ since $\min\left\{a - \frac{q(c+N)}{r}, b - p - \frac{p(c+N)}{r}\right\} > -N$ and $\nabla\zeta$ has compact support and vanishes on a neighborhood of 0.

If $c > \max\{c^0, c^1\}$, let $u(x) := |x|^{-\frac{c+N}{r}}(1 - \zeta(x))$ and argue as above, with obvious modifications.

(ii) By Kelvin transform (Remark 1.2), it suffices to consider $b - p \leq -N < a$. Note that $c^1 \leq -N < c^0$ and let $c \notin \left(-N, c^0\right]$. By (i), $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \not\subseteq L^q(\mathbb{R}^N; |x|^c dx)$ if $c > c^0$. If now $c \leq -N$, then $\zeta \notin L^r(\mathbb{R}^N; |x|^c dx)$ since $\zeta = 1$ on a neighborhood of 0, but $\zeta \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ because $a > -N$ and $\nabla\zeta$ has compact support and vanishes on a neighborhood of 0.

(iii) By contradiction, if $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{c^0} dx)$, then $||u||_{c^0,r} \leq C(||u||_{a,q} + ||\nabla u||_{b,p})$ for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. By rescaling and since $\frac{c^0+N}{r} = \frac{a+N}{q}$, it follows that $||u||_{c^0,r} \leq C(||u||_{a,q} + \lambda^{\frac{a+N}{q} - \frac{b-p+N}{p}}||\nabla u||_{b,p})$ for the same constant C independent of $\lambda > 0$. Since $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, this yields $||u||_{c^0,r} \leq C||u||_{a,q}$. In particular, if $u(x) := |x|^{-\frac{c^0+N-1}{r}} g(|x|) = |x|^{\frac{1}{r} - \frac{a+N}{q}} g(|x|)$ with $g \in C^\infty_0(0,\infty)$, it follows that $||g||_r \leq C||g||_{\frac{q}{r}-1,q}$ when $g \in C^\infty_0(0,\infty), g \geq 0$, or g is the a.e. limit of a nondecreasing sequence of such functions. Thus, a counterexample is obtained by choosing $g := \chi_{(n,n+1)}$ if $r > q$ and $g := t^{\frac{1}{n} - \frac{1}{r}}\chi_{(0,1)}$ if $r < q$ and by letting n tend to ∞.

(iv) The scaling used in (iii) now shows that if $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{c^1} dx)$, then $||u||_{c^1,r} \leq C||\nabla u||_{b,p}$ for some constant $C > 0$. The proof that C does not exist is slightly different when $a \neq -N$ and when $a = -N$.

Case (iv-1): $a \neq -N$.

By Kelvin transform, we may assume $a < -N$ with no loss of generality. It suffices to prove that, given $C > 0$,

$$||f||_{c^1+N-1,r} \leq C||f'||_{b+N-1,p}, \tag{2.1}$$

cannot hold for every $f \in W^{1,p}_{loc}(0,\infty)$ with $f \geq 0, f = 0$ on a neighborhood of 0 and $f = M$ (constant) on a neighborhood of ∞ (if so, $u(x) = f(|x|)$ is in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ irrespective of $b \in \mathbb{R}$ and of $p \geq 1, q > 0$).

It is well-known that if $1 \leq r < p$ and $C > 0$, the weighted Hardy inequality $\left(\int_0^\infty t^{\frac{r(b-p+N)}{p}-1}\left(\int_0^t g(\tau)d\tau\right)^r dt\right)^{\frac{1}{r}} \leq C\left(\int_0^\infty t^{b+N-1}g(t)^p dt\right)^{\frac{1}{p}}$ does not hold for every measurable $g \geq 0$ on $(0,\infty)$, because power weights never satisfy the necessary compatibility condition when $r < p$ ([17, Theorem 1, p. 47]). This is also true, but more delicate, when $0 < r < 1$ ([26], [27]). Thus, if $0 < r < p$, there is a sequence $g_n \geq 0$ such that $\int_0^\infty t^{b+N-1}g_n(t)^p dt < \infty$ and that

$$\left(\int_0^\infty t^{\frac{r(b-p+N)}{p}-1}\left(\int_0^t g_n(\tau)d\tau\right)^r dt\right)^{\frac{1}{r}} > n\left(\int_0^\infty t^{b+N-1}g_n^p(t)dt\right)^{\frac{1}{p}}.$$

If $b - p \geq -N$, the left-hand side is even ∞ when $g_n \neq 0$, so it may be assumed that $b - p < -N$ whenever convenient (which happens to be the case when $p = 1$). The simple proof by Sinnamon and Stepanov ([27, Theorem 2.4] if $p > 1$, [27, Theorem 3.3] if $p = 1$) reveals at once that g_n may be chosen in $L^p(0,\infty)$ and with compact support. Then, $f_n(t) := \int_0^t g_n(\tau)d\tau \geq 0$ vanishes on a neighborhood of

0 and is eventually constant. Since $\frac{r(b-p+N)}{p} - 1 = c^1 + N - 1$, this provides a counterexample to (2.1).

Case (iv-2): $a = -N$.

Then, $b-p \neq -N$ since $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. By the usual Kelvin transform argument -which does not affect $a = -N$- we may assume $b-p < -N$. It suffices to show that (2.1) cannot hold for every $f \in W^{1,p}_{loc}(0,\infty)$ with $f \geq 0$, $f = 0$ on a neighborhood of 0 and $f(t) = Mt^{-\varepsilon}$ for some constants $M, \varepsilon > 0$ and large t (if so, $u(x) = f(|x|)$ is in $W^{1,(q,p)}_{\{-N,b\}}(\mathbb{R}^N_*)$ since $b-p < -N$).

With f_n and $g_n = f_n'$ as in Case (iv-1) above, set $h_n(t) := f_n(t)$ if $0 < t < 1$ and $h_n(t) := t^{-\varepsilon_n} f_n(t)$ if $t \geq 1$, where $\varepsilon_n > 0$ will be chosen shortly. Note that $h_n = 0$ on a neighborhood of 0 and $h_n(t) = M_n t^{-\varepsilon_n}$ for $t > 0$ large enough since $f_n(t) = M_n$ is constant for large t. Since f_n provides a counterexample to (2.1) and $h_n = f_n$ on $(0,1)$, h_n will also be a counterexample if, when n is fixed, $\varepsilon_n > 0$ can be chosen so that $\int_1^\infty t^{c^1+N-1} h_n(t)^r dt$ is arbitrarily close to $\int_1^\infty t^{c^1+N-1} f_n(t)^r dt$ and $\int_1^\infty t^{b+N-1} |h_n'(t)|^p dt$ is arbitrarily close to $\int_1^\infty t^{b+N-1} |f_n'(t)|^p dt$.

By the monotone convergence of $\int_1^\infty t^{c^1+N-1-r\varepsilon} f_n(t)^r dt$ as $\varepsilon \searrow 0$, the former property holds. For the latter, it suffices to use (1) $\lim_{\varepsilon\to 0} \int_1^\infty t^{b+N-1-p\varepsilon} g_n(t)^p dt = \int_1^\infty t^{b+N-1} g_n(t)^p dt$, also proved by a monotone convergence argument, and (2) $\lim_{\varepsilon\to 0} \varepsilon^p \int_1^\infty t^{-p\varepsilon+b-p+N-1} f_n(t)^p dt = 0$, which follows from the boundedness of f_n and from $b-p < -N$.

(v) The main difference with the proof of parts (iii) and (iv) is that the scaling argument used there is inoperative because all the powers of λ cancel out. Let η denote the common value

$$\eta := \frac{a+N}{q} = \frac{b-p+N}{p} = \frac{c+N}{r}. \tag{2.2}$$

If $||u||_{c,r} \leq C(||u||_{a,q} + ||\nabla u||_{b,p})$ for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, the choice $u(x) := f(|x|)$ with $f \in C_0^\infty(0,\infty)$ yields $||f||_{c+N-1,r} \leq C(||f'||_{b+N-1,p} + ||f||_{a+N-1,q})$. If now $g \in C_0^\infty(\mathbb{R})$, then $f(t) = t^{-\eta} g(\ln t)$ with η from (2.2) is in $C_0^\infty(0,\infty)$. By the change of variable $s := \ln t$, we obtain the unweighted inequality $||g||_r \leq C(||g'||_p + ||g||_q + ||g||_p)$ for every $g \in C_0^\infty(\mathbb{R})$. With $g \neq 0$ chosen once and for all and $g(t)$ replaced by $g(\lambda t)$, $\lambda > 0$, it follows that $I_1 \leq C(\lambda^{\frac{1}{p'}+\frac{1}{r}} I_2 + +\lambda^{\frac{1}{r}-\frac{1}{q}} I_3 + \lambda^{\frac{1}{r}-\frac{1}{p}} I_4)$ with $I_1, ..., I_4 > 0$ independent of λ. Since $r < \min\{p,q\}$, the right-hand side tends to 0 with λ, which is absurd.

(vi) Argue as in (v) above, just noticing that now $\eta = 0$ in (2.2), which produces the simpler $||g||_r \leq C(||g'||_p + ||g||_q)$ when $g \in C_0^\infty(\mathbb{R})$. Then, $I_1 \leq C(\lambda^{\frac{1}{p'}+\frac{1}{r}} I_2 + \lambda^{\frac{1}{r}-\frac{1}{q}} I_3)$ for $\lambda > 0$ by rescaling, which is absurd if $r < q$. □

As a corollary, we obtain that the embedding is often characterized by a multiplicative rather than additive norm inequality (see also Section 10).

Corollary 2.2. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be such that $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ if and only if c is in the closed interval with endpoints c^0 and c^1 and there is $C > 0$ such that*

$$||u||_{c,r} \leq C||\nabla u||_{b,p}^{\theta_c} ||u||_{a,q}^{1-\theta_c}, \qquad \forall u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*), \tag{2.3}$$

where θ_c is given by (1.4). The same property is true upon replacing $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ by $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and (2.3) by

$$||u||_{c,r} \leq C||\partial_\rho u||_{b,p}^{\theta_c}||u||_{a,q}^{1-\theta_c}, \qquad \forall u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}. \tag{2.4}$$

Proof. The sufficiency follows from the arithmetic-geometric inequality. We prove the necessity for $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$. Similar arguments work in the case of $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$.

Suppose then that $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$. By part (i) of Theorem 2.1, c is in the closed interval with (distinct) endpoints c^0 and c^1. Furthermore, $||u||_{c,r} \leq C(||u||_{a,q} + ||\partial_\rho u||_{b,p})$ for every $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$. In this inequality, replace $u(x)$ by $u(\lambda x)$ with $\lambda > 0$ to get

$$\begin{aligned} ||u||_{c,r} \leq C\lambda^{\frac{c+N}{r}-\frac{a+N}{q}}||u||_{a,q} + C\lambda^{\frac{c+N}{r}-\frac{b-p+N}{p}}||\partial_\rho u||_{b,p} = \\ C\lambda^{\theta_c\frac{c^1-c^0}{r}}||u||_{a,q} + C\lambda^{(1-\theta_c)\frac{c^0-c^1}{r}}||\partial_\rho u||_{b,p}. \end{aligned} \tag{2.5}$$

If $c = c^0$ ($c = c^1$), then $\theta_c = 0$ ($\theta_c = 1$), so that $||u||_{c,r} \leq C||u||_{a,q}$ ($||u||_{c,r} \leq C||\partial_\rho u||_{b,p}$), i.e., (2.4) holds, by letting λ tend to 0 or to ∞. Otherwise, (2.4) follows by minimizing the right-hand side of (2.5) for $\lambda > 0$. This changes C, which however remains independent of u even though the minimizer is of course u-dependent. (If $\theta_c \neq 0$, (2.5) shows that $u = 0$ if $\partial_\rho u = 0$, so that it is not restrictive to assume $||u||_{a,q} > 0$ and $||\partial_\rho u||_{b,p} > 0$ in the minimization step.) ☐

The next theorem gives a different necessary condition for the continuity of the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$.

Theorem 2.3. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be given.*
(i) If $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ and $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, then $\theta_c \in [0,1]$ and

$$\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}. \tag{2.6}$$

In particular, $r \leq \max\{p^*, q\}$.
(ii) If $\frac{a+N}{q} = \frac{b-p+N}{p}$ and $c = c^0$ ($= c^1$) and if $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, then $r \leq \max\{p^*, q\}$.

Proof. (i) Part (i) of Theorem 2.1 shows that $\theta_c \in [0,1]$. The next argument is taken from [6], with a minor adjustment to fit the setting of this paper. Let $\varphi \in C_0^\infty(\mathbb{R}^N), \varphi \neq 0$, be chosen once and for all. If $x_0 \in \mathbb{R}^N$ and $R := |x_0|$ is large enough, then $\varphi(\cdot + x_0) \in C_0^\infty(\mathbb{R}^N_*) \subset W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ irrespective of a, b, p and q. By using (2.3) with $u = \varphi(\cdot + x_0)$ and by letting $R \to \infty$, we get (because $\operatorname{Supp}\varphi$ is compact) $R^{\frac{c}{r}}||\varphi||_r \leq CR^{\frac{b\theta_c}{p}+\frac{a(1-\theta_c)}{q}}||\nabla\varphi||_p^{\theta_c}||\varphi||_q^{1-\theta_c}$ for large R after changing C, whence $\frac{c}{r} \leq \frac{b\theta_c}{p} + \frac{a(1-\theta_c)}{q}$. Then, (2.6) follows by adding $\frac{N}{r}$ and using (1.6).

If $p < N$ and $r > \max\{p^*, q\}$, then (2.6) cannot hold since it fails when $\theta_c = 0$ and when $\theta_c = 1$. Thus, $r \leq \max\{p^*, q\}$ is necessary.

(ii) Use the same method as in (i), but with the additive inequality $||\varphi||_{c^0,r} \leq C(||\varphi||_{a,q} + ||\nabla\varphi||_{b,p})$. This yields $R^{\frac{c^0}{r}}||\varphi||_r \leq C(R^{\frac{a}{q}}||\varphi||_q + R^{\frac{b}{p}}||\nabla\varphi||_p)$ for large $R > 0$. By (1.3), $\frac{c^0}{r} = \frac{a}{q} + \frac{N}{q} - \frac{N}{r}$ and (since $c^0 = c^1$) $\frac{b}{p} = \frac{a}{q} + \frac{N}{q} + 1 - \frac{N}{p}$, whence

$R^{\frac{N}{q}-\frac{N}{r}}||\varphi||_r \leq C(||\varphi||_q + R^{\frac{N}{q}+1-\frac{N}{p}}||\nabla\varphi||_p)$. If $r > q$, this implies $\frac{N}{q}-\frac{N}{r} \leq \frac{N}{q}+1-\frac{N}{p}$, i.e., $r \leq p^*$, so that $r \leq \max\{p^*, q\}$ in all cases. $\square$

The above proof may give the wrong impression that (2.6) arises only as a result of integrability at infinity. That this is not the case can be seen by noticing that the choice $\varphi(x|x|^{-2}+x_0)$ instead of $\varphi(x+x_0)$ also yields (2.6), while the support of $\varphi(x|x|^{-2}+x_0)$ shrinks towards 0 as $|x_0| \to \infty$.

The verification that Theorem 2.1 and Theorem 2.3 together imply that the hypotheses made in Theorem 1.1 are necessary is routine and left to the reader.

3. The spaces $\widetilde{W}^{1,p}_{loc}$ and related concepts

In this section, we develop the background material needed for the proofs of the main results of the next two sections. Let ω_N denote the volume of the unit ball of $\mathbb{R}^N$. If $u \in L^p_{loc}(\mathbb{R}^N_*)$ with $p \geq 1$, define the spherical mean of u

$$f_u(t) := (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u(t\sigma)d\sigma. \tag{3.1}$$

By Fubini's theorem in spherical coordinates, $f_u(t)$ is defined for a.e. $t > 0$ and $f_u \in L^p_{loc}(0,\infty)$. If $u \in \widetilde{W}^{1,p}_{loc}$, where

$$\widetilde{W}^{1,p}_{loc} := \{u \in L^p_{loc}(\mathbb{R}^N_*) : \partial_\rho u \in L^p_{loc}(\mathbb{R}^N_*)\}$$

and $\partial_\rho u := \nabla u \cdot \frac{x}{|x|}$, more is true:

Lemma 3.1. *If $1 \leq p < \infty$ and $u \in \widetilde{W}^{1,p}_{loc}$, then $f_u \in W^{1,p}_{loc}(0,\infty)$. Furthermore,*

$$f_u'(t) = (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} \partial_\rho u(t\sigma)d\sigma. \tag{3.2}$$

Conversely, if $f \in W^{1,p}_{loc}(0,\infty)$ and $u(x) := f(|x|)$, then $u \in \widetilde{W}^{1,p}_{loc}$ and $f_u = f, \partial_\rho u(x) = f'(|x|)$.

Proof. Let $u \in \widetilde{W}^{1,p}_{loc}$. If $\varphi \in C_0^\infty(0,\infty)$, set $\psi(x) := \varphi(|x|)$, so that $\psi \in C_0^\infty(\mathbb{R}^N_*)$ and $\partial_\rho\psi(x) = \varphi'(|x|)$. It follows that $\langle f_u', \varphi\rangle = -(N\omega_N)^{-1}\left\langle u, |x|^{1-N}\partial_\rho\psi\right\rangle = (N\omega_N)^{-1}\left\langle |x|^{1-N}\partial_\rho u, \psi\right\rangle$ (use $\nabla\cdot\left(|x|^{-N}x\right) = 0$). Since $\partial_\rho u \in L^p_{loc}(\mathbb{R}^N_*)$, this shows that $\langle f_u', \varphi\rangle = \langle f_{\partial_\rho u}, \varphi\rangle$, that is, $f_u' = f_{\partial_\rho u} \in L^p_{loc}(0,\infty)$. Thus, $f_u \in W^{1,p}_{loc}(0,\infty)$ and (3.2) holds.

Conversely, suppose that $f \in W^{1,p}_{loc}(0,\infty)$ and set $u(x) := f(|x|)$. Then, $u \in L^p_{loc}(\mathbb{R}^N_*)$ (it is continuous) and, by [14, Theorem 4.3], $\nabla u(x) = f'(|x|)\frac{x}{|x|}$ because f is locally absolutely continuous. Thus, $u \in W^{1,p}_{loc}(\mathbb{R}^N_*) \subset \widetilde{W}^{1,p}_{loc}$ and $f'(|x|) = \nabla u(x)\cdot\frac{x}{|x|} = \partial_\rho u(x)$. That $f_u = f$ is obvious. $\square$

If Ω is an open subset of $\mathbb{R}^N$ and $u \in W^{1,1}(\Omega)$, it is well-known that $|u| \in W^{1,1}(\Omega)$ with $\nabla|u| = (\operatorname{sgn} u)\nabla u$ (see for instance [31, p. 48] or [14, Theorem 2.2] for more general statements), where $\operatorname{sgn} u$ is defined to be 0 at points where $u = 0$. This is proved by showing that if $u \in L^1(\Omega)$ and $\partial_i u \in L^1(\Omega)$ for some index $1 \leq i \leq N$, then $\partial_i|u| \in L^1(\Omega)$ and $\partial_i|u| = (\operatorname{sgn} u)\partial_i u$, because the assumptions suffice to ensure the local absolute continuity of u on almost every line segment in Ω parallel to the x_i-axis. Since a radial derivative is just a directional derivative after passing to spherical coordinates, the same arguments show that if $u \in \widetilde{W}^{1,1}_{loc}$, then $|u| \in \widetilde{W}^{1,1}_{loc}$

and $\partial_\rho |u| = (\operatorname{sgn} u)\partial_\rho u$. (That the derivative of $u(\cdot,\sigma)$ is $\partial_\rho u(\cdot,\sigma)$ can be justified by a variant of the proof of Lemma 3.1.)

Another well-known result, usually proved by localization and mollification, is that if $u \in W^{1,p}(\Omega)$ and $u \geq 0$, then $u^p \in W^{1,1}(\Omega)$ and $\partial_i(u^p) = pu^{p-1}\partial_i u$. Not surprisingly, the proof actually requires only u and $\partial_i u$ to be in $L^p(\Omega)$, so that completely similar arguments show that if $u \in \widetilde{W}^{1,p}_{loc}$ and $u \geq 0$, then $u^p \in \widetilde{W}^{1,1}_{loc}$ and $\partial_\rho u^p = pu^{p-1}\partial_\rho u$. By combining the above, we find:

Lemma 3.2. *If $1 \leq p < \infty$ and $u \in \widetilde{W}^{1,p}_{loc}$, then $f_u \in W^{1,p}_{loc}(0,\infty)$ and f_u' is given by (3.2). Furthermore, $|u|^p \in \widetilde{W}^{1,1}_{loc}$ and $\partial_\rho(|u|^p) = p|u|^{p-1}(\operatorname{sgn} u)\partial_\rho u$, where $\operatorname{sgn} u := 0$ on $u^{-1}(0)$.*

Since $f_{|u|}$ is continuous on $(0,\infty)$ when $u \in \widetilde{W}^{1,1}_{loc}$, the following two subsets are well defined:

$$\widetilde{W}^{1,1}_{loc,-} := \{u \in \widetilde{W}^{1,1}_{loc} : \underline{\lim}_{t\to\infty} f_{|u|}(t) = 0\}, \tag{3.3}$$

$$\widetilde{W}^{1,1}_{loc,+} := \{u \in \widetilde{W}^{1,1}_{loc} : \underline{\lim}_{t\to 0^+} f_{|u|}(t) = 0\}. \tag{3.4}$$

The sets $\widetilde{W}^{1,1}_{loc,\pm}$ are not closed under addition and so are not vector spaces. They are exchanged into one another by Kelvin transform. Various other properties are collected in the next lemma.

Lemma 3.3. *The following properties hold:*
(i) If $u \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$, then $|u| \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$.
(ii) $u \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+}) \Rightarrow u_S := f_u \circ |\cdot| \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$ and $\partial_\rho u_S(x) = f_u'(|x|)$.
(iii) If $u \in \widetilde{W}^{1,1}_{loc}$ and $|x|^a|u|^q \in L^1(\mathbb{R}^N)$ for some $a \in \mathbb{R}$ and some $q \geq 1$, then $u \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$ if $a \geq -N$ $(a \leq -N)$. In particular (see (1.9)), $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \subset \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$ if $a \geq -N$ $(a \leq -N)$.
(iv) If $u \in \widetilde{W}^{1,1}_{loc}$ is radially symmetric and $|x|^a|u|^q \in L^1(\mathbb{R}^N)$ for some $a \in \mathbb{R}$ and $q > 0$, then $u \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$ if $a \geq -N$ $(a \leq -N)$ In particular, if $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is radially symmetric, then $u \in \widetilde{W}^{1,1}_{loc,-}$ $(\widetilde{W}^{1,1}_{loc,+})$ if $a \geq -N$ $(a \leq -N)$.

Proof. (i) Use Lemma 3.2 and the definitions (3.3) and (3.4).

(ii) That $u_S := f_u \circ |\cdot| \in \widetilde{W}^{1,1}_{loc}$ and $\partial_\rho u_S(x) = f_u'(|x|)$ follows from Lemma 3.1. Next, the remark that $f_{|u_S|} = |f_u| \leq f_{|u|}$ shows that if also $\underline{\lim}_{t\to\infty} f_{|u|}(t) = 0$ (or $\underline{\lim}_{t\to 0^+} f_{|u|}(t) = 0$), then $\underline{\lim}_{t\to\infty} f_{|u_S|}(t) = 0$ (or $\underline{\lim}_{t\to\infty} f_{|u_S|}(t)(t) = 0$).

(iii) Suppose $a \geq -N$ and, by contradiction, $u \notin \widetilde{W}^{1,1}_{loc,-}$. Then, $f_{|u|}(t) \geq \ell > 0$ for $t \geq T$ and large $T > 0$. Thus, by (3.1), $\ell^q \leq (f_{|u|})^q(t) \leq f_{|u|^q}(t)$ for $t \geq T$, so that $\int_{|x|\geq T} |x|^a|u|^q = N\omega_N \int_T^\infty t^{a+N-1} f_{|u|^q}(t)dt \geq N\omega_N \ell^q \int_T^\infty t^{a+N-1}dt = \infty$ since $a \geq -N$. This contradicts $|x|^a|u|^q \in L^1(\mathbb{R}^N)$. The case when $a \leq -N$ follows by Kelvin transform and. the "in particular" part is obvious.

(iv) If u is radially symmetric, then $f_{|u|^q} = (f_{|u|})^q$ for every $q > 0$, so that the contradiction argument in the proof of (iii) works when $q > 0$, not just $q \geq 1$. The "in particular" part is clear if we show that $u \in \widetilde{W}^{1,1}_{loc}$. To see this, note that $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ implies $\partial_\rho u \in L^p_{loc}(\mathbb{R}^N_*)$, which, by radial symmetry, implies $\nabla u \in L^p_{loc}(\mathbb{R}^N_*)$. Thus, $u \in W^{1,p}_{loc}(\mathbb{R}^N_*)$ ([17, p. 7]) and $W^{1,p}_{loc}(\mathbb{R}^N_*) \subset \widetilde{W}^{1,1}_{loc}$ is obvious. $\square$

If $u \in L^1_{loc}(\mathbb{R}^N_*)$ is radially symmetric, then $u(x) = f_u(|x|)$. This justifies referring to the function u_S in part (ii) of Lemma 3.3 as the "radial symmetrization" of u.

Lemma 3.4. *Let $a, b \in \mathbb{R}$ and $1 \leq p, q < \infty$ be given. If $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then:*
(i) $|u| \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and $||\,|u|\,||_{a,q} = ||u||_{a,q}$, $||\partial_\rho |u|\,||_{b,p} = ||\,\partial_\rho u\,||_{b,p}$. If also u is radially symmetric, this remains true when $0 < q < 1$.
(ii) $u_S \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and $||u_S||_{a,q} \leq ||u||_{a,q}$, $||\,\partial_\rho u_S||_{b,p} \leq ||\partial_\rho u||_{b,p}$.

Proof. (i) This follows from $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}} \subset \widetilde{W}^{1,1}_{loc}$ (see Lemma 3.3 (iv) if u is radially symmetric and $0 < q < 1$) so that $\partial_\rho |u| = (\operatorname{sgn} u)\partial_\rho u$ by Lemma 3.2.

(ii) Since $u_S(x) = f_u(|x|)$ and f_u in (3.1) is continuous, u_S is continuous and so $u_S \in L^1_{loc}(\mathbb{R}^N_*)$. By (3.1), $|u_S(x)|^q \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |u(|x|\sigma)|^q d\sigma$ since $q \geq 1$ and, by (3.2) and part (ii) of Lemma 3.3, $|\partial_\rho u_S(x)|^p \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\partial_\rho u(|x|\sigma)|^p d\sigma$ a.e. Therefore, $||u_S||_{a,q} \leq ||u||_{a,q}$ and $||\partial_\rho u_S||_{b,p} \leq ||\partial_\rho u||_{b,p}$. □

We complete this section with an inequality (Theorem 3.6) which is the basic tool for the proof of Lemmas 4.3 and 4.4 in the next section.

Lemma 3.5. *Let $f \in W^{1,1}_{loc}(0,\infty)$, $f \geq 0$ and $\gamma \in \mathbb{R}$ be given.*
(i) If $\gamma \geq 1 - N$ and $\underline{\lim}_{t\to\infty} f(t) = 0$, then

$$0 \leq t^{N-1+\gamma} f(t) \leq \int_t^\infty \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq \infty, \qquad \forall t > 0. \tag{3.5}$$

(ii) If $\gamma \leq 1 - N$ and $\underline{\lim}_{t\to 0^+} f(t) = 0$, then

$$0 \leq t^{N-1+\gamma} f(t) \leq \int_0^t \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq \infty, \qquad \forall t > 0. \tag{3.6}$$

Proof. (i) Given $t > 0$, let $T > t$ and write $f(t) = f(T) - \int_t^T f'(\tau)d\tau$. Since $\gamma \geq 1 - N$ implies $t^{N-1+\gamma} \leq \tau^{N-1+\gamma}$ when $t \leq \tau$, this yields $t^{N-1+\gamma} f(t) \leq t^{N-1+\gamma} f(T) + \int_t^T \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq t^{N-1+\gamma} f(T) + \int_t^\infty \tau^{N-1+\gamma} |f'(\tau)| d\tau$. Thus, (3.5) follows from $f \geq 0$ and from $\underline{\lim}_{T\to\infty} f(T) = 0$.

(ii) Given $t > 0$, let $0 < \varepsilon < t$ and write $f(t) = f(\varepsilon) + \int_\varepsilon^t f'(\tau)d\tau$. Since $\gamma \leq 1 - N$ implies $t^{N-1+\gamma} \leq \tau^{N-1+\gamma}$ when $t \geq \tau$, this yields $t^{N-1+\gamma} f(t) \leq t^{N-1+\gamma} f(\varepsilon) + \int_\varepsilon^t \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq t^{N-1+\gamma} f(\varepsilon) + \int_0^t \tau^{N-1+\gamma} |f'(\tau)| d\tau$. Thus, (3.6) follows from $f \geq 0$ and from $\underline{\lim}_{\varepsilon\to 0} f(\varepsilon) = 0$. □

In Theorem 3.6 below, the norm notation is only used for convenience since all the norms may actually be infinite. In practice, this simply means that in the inequalities, the finiteness of the right-hand side implies the finiteness of the left-hand side, which therefore need not be assumed separately. An alternate proof can be based on the case "$q = \infty$" of [17, Theorem 2, p.40] and Kelvin transform, but the direct argument used below is more explicit and not longer.

Theorem 3.6. *Let $\gamma \in \mathbb{R}$ and $1 \leq p < \infty$ be given. There is a constant $C > 0$ such that if $u \in \widetilde{W}^{1,1}_{loc}$ is radially symmetric and either $\gamma > 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,-}$ or $\gamma < 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,+}$, then $||\,|x|^{N-1+\gamma} u||_\infty \leq C||\,|x|^{\gamma + \frac{N}{p'}} \partial_\rho u||_p$.*
Furthermore, if $p = 1$, this inequality remains true when $\gamma = 1 - N$.

Proof. Suppose first $p = 1$ and $\gamma \geq 1 - N$ and let $u \in \widetilde{W}^{1,1}_{loc,-}$. By part (i) of Lemma 3.3 and Lemma 3.2, we may and shall assume $u \geq 0$ with no loss of generality since $|| \, |x|^\gamma \partial_\rho u||_1$ and $|| \, |x|^{N-1+\gamma} u||_\infty$ are unchanged when u is replaced by $|u|$.

By Lemma 3.1, $u(x) = f_u(|x|)$ with $f_u \in \widetilde{W}^{1,1}_{loc}(0,\infty)$, $f_u \geq 0$ and $\underline{\lim}_{t\to\infty} f_{|u|}(t) = 0$ by (3.3). Thus, $|| \, |x|^{N-1+\gamma} u||_\infty = \sup_{t>0} t^{N-1+\gamma} f_u(t)$ and $\| \, |x|^\gamma \partial_\rho u\|_1 = \int_0^\infty \tau^{N-1+\gamma} |f_u'(\tau)| d\tau$ since $f_u'(|x|) = \partial_\rho u(x)$ (use $u = u_S$ and Lemma 3.3 (ii)). Hence, it suffices to show that $t^{N-1+\gamma} f_u(t) \leq \int_0^\infty \tau^{N-1+\gamma} |f_u'(\tau)| d\tau \leq \infty$ for every $t > 0$, which follows at once from (3.5) for $f = f_u$. If $\gamma \leq 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,+}$, use (3.6) instead of (3.5).

Now, let $1 < p < \infty$. Once again we assume $u \geq 0$ with no loss of generality, so that $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0,\infty)$ and $f_u \geq 0$. It suffices to prove

$$t^{N-1+\gamma} f_u(t) \leq C \int_0^\infty |f_u'(\tau)|^p \tau^{pN+p\gamma-1} d\tau, \tag{3.7}$$

for every $t > 0$. We merely show how the proof when $p = 1$ above can be modified to yield this inequality.

Suppose $\gamma > 1 - N$ and let $u \in \widetilde{W}^{1,1}_{loc,-}$. The inequality (3.5) with $\gamma = 1 - N$ -which is allowed in Lemma 3.5 - and $f = f_u$ yields $f_u(t) \leq \int_t^\infty |f_u'(\tau)| d\tau$ for every $t > 0$. Write $|f_u'(\tau)| = \left(|f_u'(\tau)| \tau^{N-1+\gamma+\frac{1}{p'}}\right) \tau^{1-N-\gamma-\frac{1}{p'}}$ and, since $\gamma > 1 - N$, use Hölder's inequality to get $f_u(t) \leq C t^{1-N-\gamma} \left(\int_t^\infty |f_u'(\tau)|^p \tau^{pN+p\gamma-1} d\tau\right)^{\frac{1}{p}}$ with $C := [p'(\gamma + N - 1)]^{-\frac{1}{p'}}$, which is stronger than (3.7). If $\gamma < 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,+}$, follow the same procedure, but starting with the inequality (3.6). □

4. Embedding theorem for radially symmetric functions

In this section, we give necessary and sufficient conditions for the continuity of the embedding of the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions into $L^r(\mathbb{R}^N; |x|^c dx)$. In principle, this can of course be done by reduction to the half-line, which is reflected in the proofs, but we have found no expository or technical advantage in doing so explicitly. Our first task will be to make sure that the cut-off operation is continuous. As a preamble, we need:

Lemma 4.1. *Let Ω denote a bounded open annulus centered at $0 \notin \overline{\Omega}$ and let $a, b \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q < \infty$ be given. There is a constant $C > 0$ such that $||u||_{p,\Omega} \leq C||u||_{\{a,b\},(q,p)}$ for every radially symmetric $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$.*

Proof. Let $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ be radially symmetric. We already pointed out in the Introduction that $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ have the same radially symmetric functions, with the same induced (quasi) norms. Since $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ implies $\nabla u \in L^p_{loc}(\mathbb{R}^N_*)$, it follows that $u \in W^{1,p}_{loc}(\mathbb{R}^N_*)$ (this was already used in the proof of Lemma 3.3 (iv)) and hence that $u \in W^{1,p}(\Omega)$. Thus, it suffices to prove that $||v||_{p,\Omega} \leq C(||v||_{q,\Omega} + ||\nabla v||_{p,\Omega})$ for every $v \in W^{1,p}(\Omega)$.

This is common knowledge when $q \geq 1$, but since only $q > 0$ is assumed, we give a proof for completeness. By contradiction, assume that there is a sequence $(v_n) \subset W^{1,p}(\Omega)$ such that $||v_n||_{p,\Omega} = 1$ and $\lim_{n\to\infty} ||v_n||_{q,\Omega} + ||\nabla v_n||_{p,\Omega} = 0$. Since (v_n) is bounded in $W^{1,p}(\Omega)$ and the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact (even

when $p = 1$), there is $v \in L^p(\Omega)$ and a subsequence, still denoted by (v_n), such that $v_n \to v$ in $L^p(\Omega)$ and that $v_n \to v$ a.e. on Ω. Obviously, $||v||_p = 1$.

Now, since $|v_n|^q \to 0$ in $L^1(\Omega)$, there is a subsequence (v_{n_k}) such that $|v_{n_k}|^q \to 0$ a.e. on Ω. Thus, $v_{n_k} \to 0$ a.e. on Ω, so that $v = 0$, which contradicts $||v||_p = 1$. $\square$

With the help of Lemma 4.1, we can now prove that truncation has the expected properties in the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions.

Lemma 4.2. *Let $a, b \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q < \infty$ be given and let $\varphi \in C^\infty(\mathbb{R}^N)$ be radially symmetric, constant on a neighborhood of 0 and constant outside a ball. Then, the multiplication by φ is continuous on the subspace of radially symmetric functions of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$.*

Proof. If $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then $||\varphi u||_{a,q} \leq ||\varphi||_\infty ||u||_{a,q}$ and $\partial_\rho(\varphi u) = \varphi \partial_\rho u + (\partial_\rho \varphi) u$. Clearly, $||\varphi \partial_\rho u||_{b,p} \leq ||\varphi||_\infty ||\partial_\rho u||_{b,p}$. To evaluate $||(\partial_\rho \varphi) u||_{b,p}$ when u is radially symmetric, note that $\operatorname{Supp} \partial_\rho \varphi$ is contained in a bounded open annulus Ω centered at $0 \notin \overline{\Omega}$. Thus, $||(\partial_\rho \varphi) u||_{b,p} \leq C ||\partial_\rho \varphi||_\infty ||u||_{\{a,b\},(q,p)}$ by Lemma 4.1 since $|x|^b$ is bounded on Ω. Altogether, this yields $||\varphi u||_{\{a,b\},(q,p)} \leq C ||u||_{\{a,b\},(q,p)}$. $\square$

The radial symmetry is unimportant in Lemmas 4.1 and 4.2 if $q \geq p$ or if $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is replaced by $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, but it does matter if $q < p$.

We first address the embedding when a and $b - p$ are on the same side of $-N$.

Lemma 4.3. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be given. If a and $b - p$ are on the same side of $-N$ (including $-N$), the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ in the following two cases (recall the definition of c^0 and c^1 in (1.3)):*
(i) $\frac{a+N}{q} \neq \frac{b-p+N}{p}, r \leq q$ and c is in the open interval with endpoints c^0 and c^1.
(ii) $\frac{a+N}{q} \neq \frac{b-p+N}{p}, b - p \neq -N$ if $p > 1, r > q$ and c is in the semi-open interval with endpoints $c^* := \left(1 - \frac{q}{r}\right) c^1 + \frac{q}{r} c^0$ (included) and c^1 (not included).

Proof. By Kelvin transform (Remark 1.2), we may assume $a \geq -N$ and $b-p \geq -N$ and, by Lemma 3.4, $u \geq 0$. By Lemma 4.2 and with ζ as in subsection 1.1, it suffices to show that $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ and that $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ for some constant $C > 0$ independent of u.

(i) The assumption $0 < r \leq q$ is retained.

Case (i-1): $b - p > -N$ or $p = 1$ and $b - 1 \geq -N$.

We first prove $||v||_{c,r} \leq C||v||_{\{a,b\},(q,p)}$ when $v := (1-\zeta)u \ (\geq 0)$. Given $\xi \in \mathbb{R}$ and $c \in \mathbb{R}$, write $|x|^c v^r = |x|^{-\xi} \left(|x|^{c+\xi} v^r\right)$. Since $\operatorname{Supp} v \subset \mathbb{R}^N \backslash B(0, \frac{1}{2})$ and by Hölder's inequality, $||v||^r_{c,r} = \int_{\mathbb{R}^N} |x|^c v^r \leq \left(\int_{\mathbb{R}^N \backslash B(0,\frac{1}{2})} |x|^{-k'\xi}\right)^{\frac{1}{k'}} \left(\int_{\mathbb{R}^N} |x|^{k(c+\xi)} v^{kr}\right)^{\frac{1}{k}}$, where $k > 1$ is arbitrary.

If $k'\xi > N$, then $M_{k,\xi} := \left(\int_{\mathbb{R}^N \backslash B(0,\frac{1}{2})} |x|^{-k'\xi}\right)^{\frac{1}{k'}} < \infty$ and it suffices to find a majorization of $\int_{\mathbb{R}^N} |x|^{k(c+\xi)} v^{kr}$. Split $|x|^{k(c+\xi)} v^{kr} = \left(|x|^{k(c+\xi)-a} v^{kr-q}\right) |x|^a v^q$, so that, if $kr - q > 0$, then $\int_{\mathbb{R}^N} |x|^{k(c+\xi)} v^{kr} \leq \left\| |x|^{k(c+\xi)-a} v^{kr-q} \right\|_\infty \int_{\mathbb{R}^N} |x|^a v^q = \left\| |x|^{\frac{k(c+\xi)-a}{kr-q}} v \right\|^{kr-q}_\infty ||v||^q_{a,q}$.

The next task is to majorize $\left\| |x|^{\frac{k(c+\xi)-a}{kr-q}} v \right\|_\infty$. This can be done by using Theorem 3.6, as we now explain. Suppose in addition that k and ξ are chosen so that $\frac{k(c+\xi)-a}{kr-q} = \frac{b-p+N}{p}$. By part (iii) of Lemma 3.3, $v \in \widetilde{W}^{1,1}_{loc,-}$ since $a \geq -N$. Next, if $\gamma := \frac{b}{p} - \frac{N}{p'}$, then $\gamma > 1 - N$ if $p > 1$ since $b - p > -N$ and $\gamma \geq 1 - N$ if $p = 1$ since $b - 1 \geq -N$. Thus, $\left\| |x|^{\frac{b-p+N}{p}} v \right\|_\infty \leq C ||\partial_\rho v||_{b,p} < \infty$ by Theorem 3.6. To summarize,

$$||v||_{c,r} \leq M_{k,\xi}^{\frac{1}{r}} C^{1-\frac{q}{kr}} ||\partial_\rho v||_{b,p}^{1-\frac{q}{kr}} ||v||_{a,q}^{\frac{q}{kr}}, \tag{4.1}$$

if k and $\xi \in \mathbb{R}$ can be found such that $k'\xi > N, kr - q > 0$ (hence $k > 1$ since $r \leq q$) and $\frac{k(c+\xi)-a}{kr-q} = \frac{b-p+N}{p}$. By introducing $s := kr - q > 0$, so that $k = \frac{s+q}{r}$, it follows that $\frac{k(c+\xi)-a}{kr-q} = \frac{b-p+N}{p}$ if and only if $\xi = \frac{arp+rs(b-p+N)}{p(s+q),} - c$ and then $k'\xi > N$ if and only if

$$c < \frac{arp + rs(b-p+N) - Nps - Npq + Npr}{p(s+q)} = \frac{c^1 s + c^0 q}{s+q}. \tag{4.2}$$

Thus, this inequality for some $s > 0$ ensures that (4.1) holds with $k := \frac{s+q}{r} > 1$ and $\xi = \frac{arp+rs(b-p+N)}{p(s+q)} - c$. The right-hand side of (4.2) is a monotone function of $s > 0$ with limits c^0 and c^1 as s tends to 0 and ∞, respectively. Therefore, $s > 0$ can be chosen so that (4.2) holds if and only if $c < \max\left\{c^0, c^1\right\}$ and then, since $v = (1-\zeta)u$ in (4.1), the arithmetic-geometric inequality yields $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ with $C > 0$ independent of u.

If now $v := \zeta u$, then once again $v \in \widetilde{W}^{1,1}_{loc,-}$ because v has bounded support. The same procedure, but with $k'\xi > N$ replaced by $k'\xi < N$, shows that $||v||_{c,r} = ||\zeta u||_{c,r} \leq C||u||_{\{a,b\},(q,p)}$ if $c > \min\left\{c^0, c^1\right\}$. Hence, both $||(1-\zeta)u||_{c,r} \leq C||u||_{\{a,b\},(q,p)}$ and $||\zeta u||_{c,r} \leq C||u||_{\{a,b\},(q,p)}$ hold when c is in the open interval with endpoints c^0 and c^1.

Case (i-2): $b - p = -N$ (and[3] $p > 1$).

If so, $\frac{a+N}{q} \neq \frac{b-p+N}{p} = 0$ and $a \geq -N$ imply $a > -N$ and $-N = c^1 < c < c^0$. If $v := (1-\zeta)u$, then, $||v||_{c,r} \leq C||v||_{a,q} \leq C||v||_{\{a,b\},(q,p)}$ by Hölder's inequality (use $|x|^c|v|^r = |x|^{c-\frac{ar}{q}} \left(|x|^{\frac{ar}{q}}(1-\zeta)^r|u|^r\right)$, $\mathrm{Supp}(1-\zeta) \subset \mathbb{R}^N \backslash B(0, \frac{1}{2})$ and $\frac{cq-ar}{q-r} < -N$, i.e., $c < c^0$, if $r < q$, or $c < a = c^0$ if $r = q$).

Next, choose $\hat{b} > b$ (so that $\hat{b} - p > -N$) such that $\hat{c}^1 := \frac{r(\hat{b}-p+N)}{p} - N < c$ and use Case (i-1) with b replaced by $\hat{b}$ -which changes c^1 into $\hat{c}^1$ but does not change c^0- and u replaced by ζu. This yields $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,\hat{b}\},(q,p)} \leq C||\zeta u||_{\{a,b\},(q,p)}$ where the second inequality follows from $\hat{b} > b$ and $\mathrm{Supp}\,\zeta \subset \overline{B}(0,1)$ (so that $||\nabla(\zeta u)||_{\hat{b},p} \leq ||\nabla(\zeta u)||_{b,p}$).

(ii) The assumption $0 < q < r$ is retained.

By part (iv) of Lemma 3.3 and Lemma 4.2, $u, \zeta u$ and $(1-\zeta)u$ are in $\widetilde{W}^{1,1}_{loc}$ (even $\widetilde{W}^{1,1}_{loc,-}$ since $a \geq -N$ and ζu has bounded support) due to radial symmetry, even when $q < 1$. Since $b - p \geq -N$ and $b - p \neq -N$ when $p > 1$, it follows that $b - p > -N$ if $p > 1$.

[3]The argument also works when $p = 1$.

The general procedure is the same as in Case (i-1), with the following difference: To prove (4.1) with $v := (1-\zeta)u$ (≥ 0), k and $\xi \in \mathbb{R}$ must be found so that $k'\xi > N, k > 1$ and $\frac{k(c+\xi)-a}{kr-q} = \frac{b-p+N}{p}$. With the same change of variable $k := \frac{s+q}{r}$ as before, $k > 1$ amounts to $s > r - q$, so that (4.1) holds for some ξ if and only if $c < \max\left\{c^*, c^1\right\}$ (the supremum of the right-hand-side of (4.2) when $s > r - q$).

Likewise, as in Case (i-1), (4.1) holds with $v = \zeta u$ if $c > \min\left\{c^*, c^1\right\}$. This proves (ii) when $b - p > -N$, or $p = 1$ and $b - 1 \geq -N$, and when c is in the *open* interval with endpoints c^* and c^1. Thus, it only remains to discuss the case $c = c^*$.

This can be done by proving the inequality (4.1) for $v = u$ radially symmetric, with $k = 1$ and $\xi = 0$ (no need to split u). Specifically, since $r > q$ (unlike in part (i)), write $||u||^r_{c^*,r} = \int_{\mathbb{R}^N} |x|^{c^*}|u|^r = \int_{\mathbb{R}^N} |x|^{a+(r-q)\left(\frac{b-p+N}{p}\right)}|u|^r \leq \left\| |x|^{\frac{b-p+N}{p}} u\right\|^{r-q}_\infty ||u||^q_{a,q}$ and notice $\left\||x|^{\frac{b-p+N}{p}}u\right\|_\infty \leq C||\partial_\rho u||_{b,p}$ by using, as before, Theorem 3.6 with $\gamma := \frac{b}{p} - \frac{N}{p'}$. This requires $b - p > -N$ if $p > 1$, but $b - 1 = -N$ is allowed if $p = 1$. $\square$

Part (ii) of Lemma 4.3 is not optimal, but before improving it (in Lemma 4.6 below) we prove a similar result when a and $b - p$ are on opposite sides of $-N$.

Lemma 4.4. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be given. If a and $b - p$ are strictly on opposite sides of $-N$, the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ in the following two cases:*
(i) $r \leq q$ and c is in the open interval with endpoints c^0 and $-N$.
(ii) $q < r, 1 - \frac{q}{r} < \theta_{-N}$ and[4] c is in the semi-open interval with endpoints $c^* := \left(1 - \frac{q}{r}\right)c^1 + \frac{q}{r}c^0$ (included) and $-N$ (not included).

Proof. Since a and $b-p$ are strictly on opposite sides of $-N$, we may assume that $b - p < -N < a$ by the usual Kelvin transform argument.

(i) By (1.3), $c^1 < -N < c^0$. Let $c \in \left(-N, c^0\right)$ be given. As in the proof of Lemma 4.3, it suffices to show that $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ and that $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ when u is radially symmetric.

Since $\mathrm{Supp}(1-\zeta) \subset \mathbb{R}^N \backslash B(0, \frac{1}{2})$, it follows that $(1-\zeta)u \in \widetilde{W}^{1,1}_{loc,+}$. As a result, the argument of the proof of Case (i-1) of Lemma 4.3, based on Theorem 3.6, can be repeated verbatim with now $\gamma := \frac{b}{p} - \frac{N}{p'} < 1 - N$. This shows that $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ since $c < \max\left\{c^0, c^1\right\} = c^0$.

The inequality $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ cannot be obtained as in Case (i-1) of Lemma 4.3 because $b - p < -N$ but $\zeta u \notin \widetilde{W}^{1,1}_{loc,+}$, so that Theorem 3.6 is not applicable. However, it can be proved with the trick used in Case (i-2) of that lemma: Since $-N < c < c^0$, part (i) of Lemma 4.3 can be used with b replaced by $p - N > b$ because $a \neq -N$ and c^1 becomes $-N$ when b is replaced by $p - N$ while c^0 is unchanged. Thus, $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,p-N\},(q,p)}$ while $||\zeta u||_{\{a,p-N\},(q,p)} \leq ||\zeta u||_{\{a,b\},(q,p)}$ since $p - N > b$ and $\mathrm{Supp}\,\zeta \subset \overline{B}(0,1)$.

(ii) Observe that $c^1 < c^* < c^0$ because $q < r$ and $c^1 < c^0$ (recall $b-p < -N < a$), while $1 - \frac{q}{r} < \theta_{-N}$ ensures that $-N < c^*$.

Let then $c \in (-N, c^*)$ be given. By using once again the fact that $(1-\zeta)u \in \widetilde{W}^{1,1}_{loc,+}$ since $\mathrm{Supp}(1-\zeta) \subset \mathbb{R}^N \backslash B(0, \frac{1}{2})$ and Theorem 3.6 with $\gamma := \frac{b}{p} - \frac{N}{p'} < 1 - N$,

[4] Since $-N$ is between c_0 and c_1 when a and $b - p$ are on opposite sides of $-N$, it follows that $\theta_{-N} \in (0, 1)$.

the argument of the proof of part (ii) of Lemma 4.3 (with obvious modifications) yields $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ because $c < c^* = \max\left\{c^*, c^1\right\}$.

If $c = c^*$, the same argument works with "$k = 1, \xi = 0$": Let $v := (1-\zeta)u \in \widetilde{W}^{1,1}_{loc,+}$ and write $||v||^r_{c^*,r} = \int_{\mathbb{R}^N} |x|^{c^*}|v|^r = \int_{\mathbb{R}^N} |x|^{a+(r-q)\left(\frac{b-p+N}{p}\right)}|v|^r \leq \left\| |x|^{\frac{b-p+N}{p}} v \right\|^{r-q}_{\infty} ||v||^q_{a,q}$. Then, use Theorem 3.6 with $\gamma := \frac{b}{p} - \frac{N}{p'} < 1 - N$ to get $\left\| |x|^{\frac{b-p+N}{p}} v \right\|_{\infty} \leq C||\partial_\rho v||_{b,p}$.

The proof of $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ when $c \in (-N, c^*]$ proceeds as in (i) above, with minor modifications. If $\hat{b} > b$, then $\hat{c}^1 := \frac{r(\hat{b}-p+N)}{p} - N > c^1$ and so $\hat{c}^* := \left(1 - \frac{q}{r}\right)\hat{c}^1 + \frac{q}{r}c^0 > c^*$. Note also that $\hat{c}^1$ is arbitrarily close to $-N$ if $\hat{b}$ is close enough to $p - N$. As a result, c is in the open interval with endpoints $\hat{c}^*$ and $\hat{c}^1$ (even when $c = c^*$) provided that $\hat{b} > p - N$ is close to $p - N$, while a and $\hat{b} - p$ are both on the right of $-N$. Thus, part (ii) of Lemma 4.3 is applicable with b replaced by $\hat{b}$ (unlike in (i), $\hat{b} = p - N$ cannot be chosen if $p > 1$ due to the requirement $\hat{b} - p \neq -N$ to use part (ii) of Lemma 4.3). □

We shall now prove optimal variants of Lemmas 4.3 and 4.4. To do this, we need a complement of part (i) of Lemma 3.4 in the radially symmetric case.

Lemma 4.5. *Let $a, b \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q < \infty$ be given. If $1 \leq \xi \leq \frac{q}{p'} + 1$ and $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is radially symmetric, then $|u|^\xi \in \widetilde{W}^{1,(q_\xi,p_\xi)}_{\{a,b_\xi\}}$, where*

$$p_\xi := \frac{pq}{p(\xi-1)+q} \geq 1,\ q_\xi := \frac{q}{\xi} > 0 \text{ and } b_\xi := \left(\frac{a(\xi-1)}{q} + \frac{b}{p}\right)p_\xi. \tag{4.3}$$

Furthermore, $|u|^\xi$ (is radially symmetric and)

$$||\,|u|^\xi||_{a,q_\xi} = ||u||^\xi_{a,q}, \qquad ||\partial_\rho(|u|^\xi)||_{b_\xi,p_\xi} \leq \xi||u||^{\xi-1}_{a,q}||\partial_\rho u||_{b,p}. \tag{4.4}$$

Proof. If $\xi = 1$, then $q_\xi = q, p_\xi = p$ and $b_\xi = b$, the case covered by Lemma 3.4, which also shows that it is not restrictive to assume $u \geq 0$. From now on, $\xi > 1$. The assumption $\xi \leq \frac{q}{p'} + 1$ ensures that $p_\xi \geq 1$ in (4.3).

That u^ξ is radially symmetric, $u^\xi \in L^{q_\xi}(\mathbb{R}^N; |x|^a dx)$ and $||\,|u|^\xi||_{a,q_\xi} = ||u||^\xi_{a,q}$ is obvious. It remains to prove that $u^\xi \in L^1_{loc}(\mathbb{R}^N_*)$, that $\partial_\rho(u^\xi) \in L^{p_\xi}(\mathbb{R}^N; |x|^{b_\xi}dx)$ and that the second inequality holds in (4.4).

By part (iv) of Lemma 3.3, $u \in \widetilde{W}^{1,1}_{loc,\pm} \subset \widetilde{W}^{1,1}_{loc}$ (depending upon whether $a \geq -N$ or $a \leq -N$). Thus, from Lemma 3.1, $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0,\infty), f_u \geq 0$ and $\partial_\rho u(x) = f'_u(|x|)$. Since $\xi > 1$, it is clear that $f^\xi_u \in W^{1,1}_{loc}(0,\infty)$ and that $\left(f^\xi_u\right)' = \xi f^{\xi-1}_u f'_u$. Hence, once again by Lemma 3.1, $u^\xi(x) = f^\xi_u(|x|)$ is in $\widetilde{W}^{1,1}_{loc} \subset L^1_{loc}(\mathbb{R}^N_*)$ and $\partial_\rho(u^\xi)(x) = \xi f^{\xi-1}_u(|x|)f'_u(|x|)$, i.e., $\partial_\rho(u^\xi) = \xi u^{\xi-1}\partial_\rho u$.

In general, if $\mu, \nu > 0$, the multiplication maps $L^\mu \times L^\nu$ into $L^{\frac{\mu\nu}{\mu+\nu}}$ and $||vw||_{\frac{\mu\nu}{\mu+\nu}} \leq ||v||_\mu ||w||_\nu$. This does not require $\mu \geq 1$ or $\nu \geq 1$ (just use $|v|^{\frac{\mu\nu}{\mu+\nu}} \in L^{1+\frac{\mu}{\nu}}$ and $|w|^{\frac{\mu\nu}{\mu+\nu}} \in L^{1+\frac{\nu}{\mu}}$ and Hölder's inequality). Now, $|x|^{\frac{a(\xi-1)}{q}}|u|^{\xi-1} \in L^{\frac{q}{\xi-1}}(\mathbb{R}^N)$ since $|x|^a|u|^q \in L^1(\mathbb{R}^N)$ and $\xi > 1$, and $|x|^{\frac{b}{p}}\partial_\rho u \in L^p(\mathbb{R}^N)$. Therefore, $|x|^{\frac{b_\xi}{p_\xi}}u^{\xi-1}\partial_\rho u \in L^{p_\xi}(\mathbb{R}^N)$ with p_ξ and b_ξ given by (4.3) and

$$\left\| |x|^{\frac{b_\xi}{p_\xi}} u^{\xi-1}\partial_\rho u \right\|_{p_\xi} \leq \left\| |x|^{\frac{a(\xi-1)}{q}} |u|^{\xi-1} \right\|_{\frac{q}{\xi-1}} ||\,|x|^{\frac{b}{p}}\partial_\rho u||_p = ||u||^{\xi-1}_{a,q}||\partial_\rho u||_{b,p}.$$

From the above, this implies $\partial_\rho(u^\xi) \in L^{p_\xi}(\mathbb{R}^N;|x|^{b_\xi}dx)$ with $||\partial_\rho(u^\xi)||_{b_\xi,p_\xi} \leq \xi||u||_{a,q}^{\xi-1}||\partial_\rho u||_{b,p}$. $\square$

Remark 4.1. *If $1 \leq \xi \leq \min\left\{q, \frac{q}{p'}+1\right\}$, Lemma 4.5 is true without the radial symmetry assumption. Indeed, if $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then $|u|^\xi \in L^{q_\xi}(\mathbb{R}^N;|x|^a dx) \subset L^1_{loc}(\mathbb{R}^N_*)$ and $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}} \subset \widetilde{W}^{1,1}$ implies that $|u|^\xi$ is locally absolutely continuous on almost every ray through the origin (see Section 3) with $\partial_\rho(|u|^\xi) = \xi|u|^{\xi-1}(\operatorname{sgn} u)\partial_\rho u \in L^{p_\xi}(\mathbb{R}^N;|x|^{b_\xi}dx) \subset L^1_{loc}(\mathbb{R}^N_*)$. This will be used elsewhere.*

Lemma 4.6. *Let $a,b,c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q,r < \infty$ be given and let $\breve{\theta} := \left(1-\frac{q}{r}\right)\left(\frac{q}{p'}+1\right)^{-1} < 1$ (≤ 0 if $r \leq q$). The subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N;|x|^c dx)$ in the following two cases.*
(i) a and $b-p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is in the open interval with endpoints c^0 and c^1 and $\theta_c \geq \breve{\theta}$ (vacuous if $r \leq q$).
(ii) a and $b-p$ are strictly on opposite sides of $-N, c$ is in the open interval with endpoints c^0 and $-N$ and $\theta_c \geq \breve{\theta}$ (empty set if $\breve{\theta} \geq \theta_{-N}$).

Proof. If $r \leq q$ (so that $\breve{\theta} \leq 0$) or if $r > q$ and $p = 1$ (so that $\breve{\theta} = 1 - \frac{q}{r}$), (i) follows from Lemma 4.3 (where $b-p \neq -N$ is not required in part (ii) when $p=1$) and (ii) follows from Lemma 4.4. From now on, $r > q$ (so that $\breve{\theta} \in (0,1)$) and $p > 1$. For convenience, we set $\breve{\xi} := \frac{q}{p'}+1 > 1$. In particular, the interval $(1,\breve{\xi}]$ is not empty, which is implicitly used below.

(i) Let $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ be radially symmetric. If $1 \leq \xi \leq \breve{\xi}$, then, by Lemma 4.5, $|u|^\xi \in \widetilde{W}^{1,(q_\xi,p_\xi)}_{\{a,b_\xi\}}$ with $q_\xi > 0, p_\xi \geq 1$ and b_ξ given by (4.3). A routine verification shows that a and $b_\xi - p_\xi$ are on the same side of $-N$ (since the same thing is true of a and $b-p$) and that $\frac{a+N}{q_\xi} \neq \frac{b_\xi-p_\xi+N}{p_\xi}$ (since $\frac{a+N}{q} \neq \frac{b-p+N}{p}$).

Furthermore, $b_\xi - p_\xi \neq -N$ if $\xi > 1$ (which need not be true if $\xi = 1$ since $b-p \neq -N$ is *not* assumed). Indeed, $b_\xi - p_\xi = -N$ amounts to $\frac{a+N}{q}(\xi-1) + \frac{b-p+N}{p} = 0$. Since a and $b-p$ are on the same side of $-N$, this can only happen if $a+N = b-p+N = 0$ when $\xi > 1$, which contradicts the assumption $\frac{a+N}{q} \neq \frac{b-p+N}{p}$.

Accordingly, from part (ii) of Lemma 4.3 with b,p,q and r replaced by b_ξ,p_ξ,q_ξ and s, respectively, $W^{1,(q_\xi,p_\xi)}_{\{a,b_\xi\}}(\mathbb{R}^N_*) \hookrightarrow L^s(\mathbb{R}^N;|x|^c dx)$ whenever $1 < \xi \leq \breve{\xi}, 0 < q_\xi < s$ and c is in the semi-open interval with endpoints $a + \frac{(s-q_\xi)(b_\xi-p_\xi+N)}{p_\xi}$ (included; this corresponds to c^* with the parameters b_ξ,p_ξ,q_ξ,s) and $\frac{s(b_\xi-p_\xi+N)}{p_\xi} - N$ (not included; this corresponds to c^1 with the parameters b_ξ,p_ξ,q_ξ,s). Since $r > q$, and $q_\xi = \frac{q}{\xi}$, the condition $0 < q_\xi < s$ holds when $s = \frac{r}{\xi}$. If so, the embedding inequality $||\,|u|^\xi||_{c,\frac{r}{\xi}} \leq C_\xi(||\,|u|^\xi||_{a,q_\xi} + ||\partial_\rho(|u|^\xi)||_{b_\xi,p_\xi})$ reads (use (4.4))

$$||u||^\xi_{c,r} \leq C_\xi(||u||^\xi_{a,q} + ||u||^{\xi-1}_{a,q}||\partial_\rho u||_{b,p}) \leq C_\xi(||u||_{a,q} + ||\partial_\rho u||_{b,p})^\xi,$$

so that $||u||_{c,r} \leq C_\xi^{\xi^{-1}}||u||_{\{a,b\},(q,p)}$. Above, c is in the semi-open interval J_ξ with (distinct) endpoints $e_1(\xi) := a + \frac{(r-q)(b_\xi-p_\xi+N)}{\xi p_\xi}$ (included) and $e_2(\xi) := \frac{r(b_\xi-p_\xi+N)}{\xi p_\xi} -$

N (not included) and $1 < \xi \leq \breve{\xi}$. Thus, when $c \in J := \cup_{\xi \in (1,\breve{\xi}]} J_\xi$,

$$||u||_{c,r} \leq C(||u||_{a,q} + ||\partial_\rho u||_{b,p}), \tag{4.5}$$

for some constant $C > 0$ independent of the radially symmetric $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ (specifically, $C = C_\xi^{\xi^{-1}}$ for any ξ such that $c \in J_\xi$).

Since the distinct endpoints of J_ξ depend continuously upon ξ, the lower (upper) endpoint $e_-(\xi)$ ($e_+(\xi)$) is either $e_1(\xi)$ for every ξ or $e_2(\xi)$ for every ξ. Hence, $e_\pm$ are continuous and never equal functions of ξ. With that remark, it is an easy exercise to show that J contains the open interval with endpoints $\inf e_-$ and $\sup e_+$.

If $\frac{a+N}{q} > \frac{b-p+N}{p}$, then $e_1 > e_2$ and both e_1 and e_2 are increasing functions of ξ, so that J contains $(e_2(1), e_1(\breve{\xi}))$. In addition, since it contains $e_1(\breve{\xi}) \in J_{\breve{\xi}}$, it contains -and, in fact, coincides with- $(e_2(1), e_1(\breve{\xi})]$.

If $\frac{a+N}{q} < \frac{b-p+N}{p}$, then $e_2 > e_1$ and both e_1 and e_2 are decreasing functions of ξ, so that J contains the open interval $(e_1(\breve{\xi}), e_2(1))$. Once again, it also contains $e_1(\breve{\xi})$. Therefore, in all cases, J is the semi-open interval with endpoints $e_1(\breve{\xi}) = \breve{\theta}c^1 + (1-\breve{\theta})c^0$ (included) and $e_2(1) = c^1$ (not included). For every c in that interval, (4.5) holds for some constant C independent of the radially symmetric $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$. Clearly, J is equally characterized as the set of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c \geq \breve{\theta}$.

(ii) First, since a and $b - p$ are on opposite sides of $-N$, it is obvious that $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. The proof will proceed as in part (i), but extra technicalities arise from the fact that the points a and $b_\xi - p_\xi$ (see (4.3)) need not remain on opposite sides of $-N$ for all $\xi \in [1, \breve{\xi}]$.

Nonetheless, since $b_\xi - p_\xi$ is a strictly monotone function of ξ equal to $b-p$ when $\xi = 1$ and since a and $b - p$ are strictly on opposite sides of $-N$, there are only two options: Either a and $b_\xi - p_\xi$ are strictly on opposite sides of $-N$ when $\xi = \breve{\xi}$ -which amounts to a and $\frac{b}{p} + \frac{a}{p'} - 1$ being strictly on opposite sides of $-N$- and then the same thing is true for every $\xi \in [1, \breve{\xi}]$, or $b_{\xi_0} - p_{\xi_0} = -N$ for some unique $\xi_0 \in (1, \breve{\xi}]$, and then a and $b_\xi - p_\xi$ are on the same side of $-N$ for every $\xi \in [\xi_0, \breve{\xi}]$.

Case (ii-1): a and $\frac{b}{p} + \frac{a}{p'} - 1$ are strictly on opposite sides of $-N$.

Replace q, r, b, p by $q_{\breve{\xi}} = \frac{q}{\breve{\xi}}, r_{\breve{\xi}} = \frac{r}{\breve{\xi}}, b_{\breve{\xi}} = \frac{a}{p'} + \frac{b}{p}, p_{\breve{\xi}} = 1$, respectively, in part (ii) of Lemma 4.4 and use that result with u replaced by $|u|^{\breve{\xi}}$. This is justified by Lemma 4.5. However, it is crucial to notice that, due to the change of parameters, the condition "$1 - \frac{q}{r} < \theta_{-N}$" in Lemma 4.4 does not involve θ_{-N} but, instead, the number $\breve{\theta}_{-N}$ given by the same formula (1.4) when c^0 and c^1 are replaced by $\breve{c}^0$ and $\breve{c}^1$ defined by (1.3) with the new parameters $q_{\breve{\xi}}, r_{\breve{\xi}}, b_{\breve{\xi}}, p_{\breve{\xi}}$. Thus, $\breve{c}^0 = c^0$ but $\breve{c}^1 = r\breve{\xi}^{-1}\left(\frac{a+N}{p'} + \frac{b-p+N}{p}\right) - N$, so that $\breve{c}^1 - \breve{c}^0 = r\breve{\xi}^{-1}\left(\frac{b-p+N}{p} - \frac{a+N}{q}\right)$. With this remark, it is readily checked that $\breve{\theta}_{-N} = \breve{\xi}\theta_{-N}$, so that the condition $1 - \frac{q_{\breve{\xi}}}{r_{\breve{\xi}}} < \breve{\theta}_{-N}$ is $\breve{\theta} := \left(1 - \frac{q}{r}\right)\left(\frac{q}{p'} + 1\right)^{-1} < \theta_{-N}$.

In summary, the continuity of the embedding is ensured if $\breve{\theta} < \theta_{-N}$ and c is in the semi-open interval with endpoints $\breve{c} := \left(1 - \frac{q}{r}\right)\breve{c}^1 + \frac{q}{r}\breve{c}^0 = \breve{\theta}c^1 + (1-\breve{\theta})c^0$

(included) and $-N$ (not included), which -since $r > q$- coincides with the set of those c in the open interval with endpoints c^0 and $-N$ such that $\theta_c \geq \breve{\theta}$.

Case (ii-2): $b_{\xi_0} - p_{\xi_0} = -N$ for some $\xi_0 \in (1, \breve{\xi})$.

Since a and $b_\xi - p_\xi$ are on the same side of $-N$ for $\xi \in [\xi_0, \breve{\xi}]$ and since $b_\xi - p_\xi \neq -N$ if $\xi \in (\xi_0, \breve{\xi}]$, part (ii) of Lemma 4.3 with u replaced by $|u|^\xi$ and q, r, b, p replaced by $\frac{q}{\xi}, \frac{r}{\xi}, b_\xi, p_\xi$, respectively, yields that the subspace of $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ for every $c \in J := \cup_{\xi \in (\xi_0, \breve{\xi}]} J_\xi$, where J_ξ is the semi-open interval with endpoints $e_1(\xi) := a + \frac{(r-q)(b_\xi - p_\xi + N)}{\xi p_\xi}$ (included) and $e_2(\xi) := \frac{r(b_\xi - p_\xi + N)}{\xi p_\xi} - N$ (not included). Both endpoints are distinct (because $\frac{a+N}{q} \neq \frac{b-p+N}{p}$) and on the same side of $-N$ when $\xi > \xi_0$. By arguing as in the proof of (i) above, J is found to be the semi-open interval with endpoints $e_1(\breve{\xi}) = \breve{c} = \breve{\theta} c^1 + (1 - \breve{\theta}) c^0$ (included) and $e_2(\xi_0) = -N$ (not included), exactly as in (ii-1). Therefore, the final argument is also the same.

Case (ii-3): $b_{\xi_0} - p_{\xi_0} = -N$ with $\xi_0 = \breve{\xi}$.

Since a and $b_{\breve{\xi}} - p_{\breve{\xi}}$ are on the same side of $-N$ *and* since $p_{\breve{\xi}} = 1$, it suffices to use part (ii) of Lemma 4.3 with u replaced by $|u|^{\breve{\xi}}$ and q, r, b, p replaced by $\frac{q}{\breve{\xi}}, \frac{r}{\breve{\xi}}, b_{\breve{\xi}} = \frac{b}{p} + \frac{a}{p'}, p_{\breve{\xi}} = 1$, respectively. $\square$

It is informative that even if $q, r \geq 1$ in Lemma 4.6, the proof involves part (ii) of Lemmas 4.3 and 4.4 when $q, r > 0$ ($q, r \geq 1$ is not enough). The next theorem gives necessary and sufficient conditions for the continuous embedding of the subspace of radially symmetric functions.

Theorem 4.7. *Let $a, b, c \in \mathbb{R}, 1 \leq p < \infty$ and $0 < q, r < \infty$ and set $\breve{\theta} := \left(1 - \frac{q}{r}\right)\left(\frac{q}{p'} + 1\right)^{-1}$. The subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ (and hence into $\widetilde{W}^{1,(r,p)}_{\{c,b\}}$) if and only if one of the following conditions holds:*
(i) a and $b - p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is in the open interval with endpoints c^0 and c^1 and $\theta_c \geq \breve{\theta}$ (vacuous if $q \geq r$).
(ii) a and $b - p$ are strictly on opposite sides of $-N, c$ is in the open interval with endpoints c^0 and $-N$ and $\theta_c \geq \breve{\theta}$ (empty set if $\breve{\theta} \geq \theta_{-N}$).
(iii) $r \geq p, a \leq -N$ and $b - p < -N$ or $a \geq -N$ and $b - p > -N, c = c^1$. Furthermore, there is a constant $C > 0$ such that

$$||u||_{c,r} \leq C||\partial_\rho u||_{b,p}, \tag{4.6}$$

for every radially symmetric function $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$.
(iv) $r = q$ and $c = c^0$ $(= a)$, or $p \neq q, \min\{p, q\} \leq r \leq \max\{p, q\}, \frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$ and $c = c^0$ $(= c^1)$. Furthermore, there is a constant $C > 0$ such that

$$||u||_{c,r} \leq C||\partial_\rho u||^\theta_{b,p}||u||^{1-\theta}_{a,q}, \tag{4.7}$$

for every radially symmetric function $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, where $\theta = 0$ if $r = q$ and $c = a$ and $\theta = \frac{p(r-q)}{r(p-q)}$ otherwise.
(v) $a = -N, b = p - N, r > q$ and $c = c^0$ $(= c^1 = -N)$. Furthermore, there is a

constant $C > 0$ such that

$$||u||_{-N,r} \leq C||\partial_\rho u||_{p-N,p}^{\breve{\theta}}||u||_{-N,q}^{1-\breve{\theta}}, \tag{4.8}$$

for every radially symmetric function $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$.

Proof. The theorem is (as it should be) equivalent to Theorem 1.1 when $N = 1$ (in particular, $p^* = \infty$ regardless of p and $\frac{1}{p} - \frac{1}{N} - \frac{1}{q} = -\left(\frac{1}{p'} + \frac{1}{q}\right)$) and a, b, c replaced by $a + N - 1, b + N - 1$ and $c + N - 1$, respectively. Since the hypotheses of Theorem 1.1 are necessary (Section 2), the necessity follows.

The sufficiency of parts (i) and (ii) was already proved in Lemma 4.6. To complete the proof, we show that parts (iii), (iv) or (v) are also sufficient.

(iii) By Kelvin transform (Remark 1.2), we may assume $a \leq -N$ and $b-p < -N$. In particular, $u \in \widetilde{W}^{1,1}_{loc,+}$ by part (iv) of Lemma 3.3. By part (i) of the same lemma, we may also assume $u \geq 0$ with no loss of generality. Then, $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0,\infty), f_u \geq 0$ and $\underline{\lim}_{t\to 0^+} f_u(t) = 0$, so that $f_u(t) \leq \int_0^t |f_u'(\tau)|d\tau$ by (3.6) with $\gamma = 1 - N$ and $f = f_u$.

On the other hand,

$$\left(\int_0^\infty t^{\frac{r(b-p+N)}{p}-1}\left(\int_0^t |f_u'(\tau)|d\tau\right)^r dt\right)^{\frac{1}{r}} \leq C\left(\int_0^\infty t^{b+N-1}|f_u'(t)|^p dt\right)^{\frac{1}{p}},$$

by a weighted Hardy inequality of Bradley for nonnegative measurable functions on $(0,\infty)$ ([5, Theorem 1], [17, p. 40]) inspired by Muckenhoupt [19] when $r = p$. This yields (4.6) since $c = c^1 = \frac{r(b-p+N)}{p} - N$ and $\partial_\rho u(x) = f_u'(|x|)$ (Lemma 3.1).

(iv) This is trivial if $r = q$ and $c = a$. From now on, $p \neq q$ and r is between p and q (both included), so that $r = \mu p + (1-\mu)q$ where $\mu = \frac{r-q}{p-q} \in [0,1]$, whence $\mu(b-p) + (1-\mu)a = c^0 = c$ (use $b - p = \frac{p(a+N)}{q} - N$). Thus, if u is measurable, $\int_{\mathbb{R}^N} |x|^c|u|^r = \int_{\mathbb{R}^N} (|x|^{b-p}|u|^p)^\mu(|x|^a|u|^q)^{1-\mu}$ and, by Hölder's inequality,

$$||u||_{c,r}^r \leq ||u||_{b-p,p}^{\mu p}||u||_{a,q}^{(1-\mu)q}. \tag{4.9}$$

Since $\frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$, both a and $b-p$ are on the same side of $-N$ and neither equals $-N$. Therefore, when $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is radially symmetric, $||u||_{b-p,p} \leq C||\partial_\rho u||_{b,p}$ by (iii) with $r = p$ (hence $c = b - p$). By substitution into (4.9), $||u||_{c,r} \leq C||\partial_\rho u||_{b,p}^{\mu\frac{p}{r}}||u||_{a,q}^{(1-\mu)\frac{q}{r}} = C||\partial_\rho u||_{b,p}^\theta||u||_{a,q}^{1-\theta}$ with $\theta = \mu\frac{p}{r} = \frac{p(r-q)}{r(p-q)}$. This proves (4.7) and hence the embedding property as well.

(v) If $r > q, N = 1$ and $a = b = c = 0$, it follows from part (i) of the theorem if $p = 1$ and from its part (ii) if $p > 1$, that the subspace of even functions in the unweighted space $W^{1,(q,p)}(\mathbb{R}_*)$ is continuously embedded into $L^r(\mathbb{R})$. In this one-dimensional setting, this readily implies the same result without the evenness assumption, i.e., $W^{1,(q,p)}(\mathbb{R}_*) \hookrightarrow L^r(\mathbb{R})$, and then

$$||g||_r \leq C||g'||_p^{\breve{\theta}}||g||_q^{1-\breve{\theta}}, \tag{4.10}$$

for $g \in W^{1,(q,p)}(\mathbb{R}_*)$ by the usual rescaling argument. In particular, (4.10) holds with $g \in W^{1,(q,p)}(\mathbb{R})$ (if $g \in C_0^\infty(\mathbb{R})$ and $q \geq 1$, this also follows from [6]).

Now, as in (iii), if $u \in \widetilde{W}^{1,(q,p)}_{\{-N,p-N\}}$ is radially symmetric, then $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0,\infty)$ and $\partial_\rho u(x) = f_u'(|x|)$, so that $||u||^q_{-N,q} = N\omega_N \int_0^\infty t^{-1}|f_u(t)|^q dt < \infty$ and $||\partial_\rho u||^p_{p-N,p} = N\omega_N \int_0^\infty t^{p-1}|f_u'(t)|^p dt < \infty$.

On the other hand, with $g(s) := f_u(e^s)$, it is readily checked that $g \in W^{1,(q,p)}(\mathbb{R})$ with $||g||^q_q = \int_0^\infty t^{-1}|f_u(t)|^q dt$ and $||g'||^p_p = \int_0^\infty t^{p-1}|f_u'(t)|^p dt$. Therefore, (4.10) may be rewritten as $||u||_{-N,r} \leq C||\partial_\rho u||^{\breve{\theta}}_{p-N,p}||u||^{1-\breve{\theta}}_{-N,q}$. This completes the proof. $\square$

Remark 4.2. *Since* $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ *and* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ *contain the same radially symmetric functions and the induced norms are the same, Theorem 4.7 is also true when* $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ *is replaced by* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$.

Remark 4.3. *In part (ii) of Theorem 4.7, the admissible interval is empty if* $\breve{\theta} \geq \theta_{-N}$*, which can only happen if* $r > q$*. However, a careful examination of the proofs reveals that the subspace of (radially symmetric) functions with support in a ball* $\overline{B}$ *centered at 0 is continuously embedded into* $L^r(\mathbb{R}^N; |x|^c dx)$ *if* $c > -N$ *when* $b-p < -N < a$ *and if* $c \geq \breve{c}$ *(even if* $\breve{c} = -N$*) when* $a < -N < b-p$*. For functions with support in* $\mathbb{R}^N \backslash B$ *the conditions (*$c \leq \breve{c}$ *if* $b-p < -N < a$ *and* $c < -N$ *if* $a < -N < b-p$*) follow by Kelvin transform. Details are left to the reader.*

5. Embedding theorem when $1 \leq r \leq \min\{p,q\}$

We now extend Theorem 4.7 to the non-symmetric case when $1 \leq r \leq \min\{p,q\}$. To do this, we need the following refinement of part (ii) of Lemma 3.4:

Lemma 5.1. *Let* $a, b \in \mathbb{R}$ *and* $1 \leq r \leq p, q < \infty$ *be given. If* $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$*, then* $v := [(|u|^r)_S]^{\frac{1}{r}} \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$*. Furthermore,* $||v||_{a,q} \leq ||u||_{a,q}$ *and* $||\,\partial_\rho v||_{b,p} \leq ||\partial_\rho u||_{b,p}$*, so that* $||v||_{\{a,b\},(q,p)} \leq ||u||_{\{a,b\},(q,p)}$.

Proof. By part (i) of Lemma 3.4, $|u| \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, so that it is not restrictive to assume $u \geq 0$. Since $v(x) = [f_{u^r}(|x|)]^{\frac{1}{r}}$ with $f_{u^r}(t) := (N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} u^r(t\sigma)d\sigma$, it follows from $r \leq q$ and Hölder's inequality that $(v(x))^q \leq (N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} u^q(|x|\sigma)d\sigma$. Thus, $||v||_{a,q} \leq ||u||_{a,q}$ is clear.

We now show that $\partial_\rho v \in L^p(\mathbb{R}^N; |x|^b dx)$ and prove the desired estimate. Formally, if $h := (f_{u^r})^{\frac{1}{r}}$, then $h' = \frac{1}{r}(f_{u^r})^{-\frac{1}{r'}} f'_{u^r}$ but, by the de la Vallée Poussin criterion ([29], [14, Lemma 1.2], [25, Corollary 8]), this formula holds and $h \in W^{1,1}_{loc}(0,\infty)$ if and only if $f_{u^r} \in W^{1,1}_{loc}(0,\infty)$ and $(f_{u^r})^{-\frac{1}{r'}} f'_{u^r} \in L^1_{loc}(0,\infty)$, with the understanding that $(f_{u^r})^{-\frac{1}{r'}} f'_{u^r} = 0$ when $f'_{u^r} = 0$, irrespective of whether $(f_{u^r})^{-\frac{1}{r'}}$ is defined. Since $f'_{u^r} = 0$ a.e. on $(f_{u^r})^{-1}(0)$, this amounts to defining $(f_{u^r})^{-\frac{1}{r'}} f'_{u^r} = 0$ on $(f_{u^r})^{-1}(0)$. That $(f_{u^r})^{-\frac{1}{r'}} f'_{u^r} \in L^1_{loc}(0,\infty)$ is verified below.

First, $u \in \widetilde{W}^{1,r}_{loc}$ since $r \leq p, q$. By Lemma 3.2, $u^r \in \widetilde{W}^{1,1}_{loc}$ (so that $f_{u^r} \in W^{1,1}_{loc}(0,\infty)$) and $\partial_\rho(u^r) = ru^{r-1}\partial_\rho u$. Upon replacing u by u^r in (3.2) and by Hölder's inequality, it follows that $|f'_{u^r}| \leq r(f_{u^r})^{\frac{1}{r'}}\left((N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} |\partial_\rho u|^r d\sigma\right)^{\frac{1}{r}} \leq r(f_{u^r})^{\frac{1}{r'}}\left((N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} |\partial_\rho u|^p d\sigma\right)^{\frac{1}{p}}$. Since $(f_{u^r})^{-\frac{1}{r'}} f'_{u^r} = 0$ on $(f_{u^r})^{-1}(0)$, this yields $(f_{u^r})^{-\frac{1}{r'}}|f'_{u^r}| \leq r\left((N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} |\partial_\rho u|^p d\sigma\right)^{\frac{1}{p}} \in L^p_{loc}(0,\infty) \subset L^1_{loc}(0,\infty)$.

From the above, $h \in W^{1,1}_{loc}(0,\infty), h' = \frac{1}{r}(f_{u^r})^{-\frac{1}{r'}} f'_{u^r}$ and, in addition, $|h'(t)| \leq \left((N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\partial_\rho u(t\sigma)|^p d\sigma\right)^{\frac{1}{p}}$. Since $\partial_\rho v(x) = h'(|x|)$ by Lemma 3.1, $|\partial_\rho v(x)|^p \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\partial_\rho u(|x|\sigma)|^p d\sigma$, so that $|| \, \partial_\rho v||_{b,p} \leq ||\partial_\rho u||_{b,p}$. □

Theorem 5.2. *Let $a, b, c \in \mathbb{R}$ and $1 \leq r \leq p, q < \infty$ be given. Then, $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \hookrightarrow \widetilde{W}^{1,(r,p)}_{\{c,b\}}$) in the following cases:*
(i) a and $b - p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ and c is in the open interval with endpoints c^0 and c^1.
(ii) a and $b - p$ are strictly on opposite sides of $-N$ (hence $\frac{a+N}{q} \neq \frac{b-p+N}{p}$) and c is in the open interval with endpoints c^0 and $-N$.
(iii) $r = q \leq p$ and $c = a$.
(iv) $r = p \leq q$, $a \leq -N$ and $b-p < -N$ or $a \geq -N$ and $b-p > -N$, and $c = b-p$.

Proof. (i)-(ii) Set $v := [(|u|^r)_S]^{\frac{1}{r}}$. By Lemma 5.1, $v \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and $||v||_{\{a,b\},(q,p)} \leq ||u||_{\{a,b\},(q,p)}$. Thus, since v is radially symmetric, it follows from parts (i) and (ii) of Theorem 4.7 (where $\theta_c \geq \breve{\theta}$ holds since $\breve{\theta} \leq 0$) that $||v||_{c,r} \leq C||u||_{\{a,b\},(q,p)}$, where $C > 0$ is independent of u. The conclusion follows from the remark that $||v||_{c,r} = ||u||_{c,r}$.

(iii) is trivial.

(iv) Argue as in (i)-(ii) above, now using part (iii) of Theorem 4.7 with $r = p$. □

When $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is replaced by the smaller space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, Theorem 5.2 coincides with Theorem 1.1 when $1 \leq r \leq p, q < \infty$. Indeed, $r \leq \min\{p, q\}$ implies $r \leq \min\{p^*, q\}$, so that the inequality $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$ holds for every c in the closed interval with endpoints c^0 and c^1 (Remark 1.1).

6. The Caffarelli-Kohn-Nirenberg Lemma and application

The reduction to the radially symmetric case in the previous section cannot be used when $r > \min\{p, q\}$. Consistent with the strategy outlined in the Introduction, this section is devoted to the formulation and proof of an embedding property for a direct complement of the subspace of radially symmetric functions.

It will be necessary to confine attention to the space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ (as opposed to $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$), because integrability conditions about all the first order partial derivatives are implicitly required. While phrased differently and under less general conditions, Lemma 6.1 below is already contained in [6].

Lemma 6.1 (CKN lemma)**.** *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be given and suppose that there are $\delta \leq \frac{b}{p}$ and $\theta \in [0, 1]$ such that:*
(i) $\frac{c}{r} = \theta\delta + (1 - \theta)\frac{a}{q}$.
(ii) $\frac{c+N}{r} = \theta\frac{b-p+N}{p} + (1 - \theta)\frac{a+N}{q}$.
(iii) $\frac{\theta r}{p} + \frac{(1-\theta)r}{q} \geq 1$.
Then,

$$W_0 := \left\{u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) : u_S = 0\right\} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx) \tag{6.1}$$

and there is a constant $C > 0$ such that

$$||u||_{c,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{(1-\theta)}, \qquad \forall u \in W_0. \tag{6.2}$$

Proof. Of course, it suffices to prove (6.2). For $\tau > 0$, let Ω_τ denote the annulus $\{x \in \mathbb{R}^N : \tau < |x| < 2\tau\}$. Under the conditions (i) and (ii) of the lemma[5], it is shown in [6, pp. 262-263] that the *unweighted* inequality

$$\int_{\Omega_1} |u|^r \leq C \left(\int_{\Omega_1} |\nabla u|^p\right)^{\frac{\theta r}{p}} \left(\int_{\Omega_1} |u|^q\right)^{\frac{(1-\theta)r}{q}}, \tag{6.3}$$

holds for some constant C and every $u \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\Omega_1} u = 0$. The proof relies on the Gagliardo-Nirenberg and Sobolev inequalities. (Since a, b, c are not involved in (6.3), what matters is the relation $\frac{1}{r} = \theta\left(\frac{1}{p} - \frac{1}{N} + \gamma\right) + (1-\theta)\frac{1}{q}$ with $\gamma \geq 0$; that $\gamma = \frac{1}{N}\left(\frac{b}{p} - \delta\right)$ from (i) and (ii) combined, is not relevant at this stage.)

From the geometric properties of Ω_1, the denseness of $C_0^\infty(\mathbb{R}^N)$ in the unweighted space $W^{1,(q,p)}(\Omega_1) := \{u \in L^q(\Omega_1) : \nabla u \in L^p(\Omega_1)\}$ is routine (see [3], [23] for more general results) and it is trivial that denseness remains true if, in both spaces, attention is confined to functions with mean 0 on Ω_1. Thus, (6.3) continues to hold for $u \in W^{1,(q,p)}(\Omega_1)$ such that $\int_{\Omega_1} u = 0$ and hence for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ such that $\int_{\Omega_1} u = 0$ since, irrespective of a and b, the restrictions to Ω_1 of functions in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ are obviously in $W^{1,(q,p)}(\Omega_1)$.

If $x \in \Omega_1$, then $|x|^a, |x|^b$ are bounded below and $|x|^c$ is bounded above. Thus, after changing C, (6.3) yields $\int_{\Omega_1} |x|^c|u|^r \leq C\left(\int_{\Omega_1} |x|^b|\nabla u|^p\right)^{\frac{\theta r}{p}} \left(\int_{\Omega_1} |x|^a|u|^q\right)^{\frac{(1-\theta)r}{q}}$ for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ such that $\int_{\Omega_1} u = 0$. By rescaling and using (ii), this implies, with the same C independent of τ,

$$\int_{\Omega_\tau} |x|^c|u|^r \leq C \left(\int_{\Omega_\tau} |x|^b|\nabla u|^p\right)^{\frac{\theta r}{p}} \left(\int_{\Omega_\tau} |x|^a|u|^q\right)^{\frac{(1-\theta)r}{q}}, \tag{6.4}$$

for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ such that $\int_{\Omega_\tau} u = 0$. In particular, (6.4) holds for every $\tau > 0$ and every $u \in W_0$ defined in (6.1).

It is also observed in [6, p. 268] that if $k \in \mathbb{Z}$ and $A_k, B_k \geq 0$ and if $\alpha, \beta \geq 0$ satisfy $\alpha + \beta \geq 1$, then

$$\sum_{k\in\mathbb{Z}} A_k^\alpha B_k^\beta \leq \left(\sum_{k\in\mathbb{Z}} A_k\right)^\alpha \left(\sum_{k\in\mathbb{Z}} B_k\right)^\beta, \tag{6.5}$$

where the first (second) factor on the right is 1 when $\alpha = 0$ ($\beta = 0$). Thus, when condition (iii) holds, (6.2) follows from (6.5) and (6.4) with $\tau = 2^k, k \in \mathbb{Z}$. □

There is a clearer and more convenient formulation of Lemma 6.1:

Corollary 6.2. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be given.*
(i) If $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ and c is in the closed interval with endpoints c^0 and c^1, then $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if the following conditions hold (with θ_c given by (1.4)):
(i-1) Either $r = q$ and $c = c^0$ $(= a)$ or $c \neq c^0$ and $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.

[5] None of the other assumptions in [6] is involved.

(i-2) $\frac{\theta_c r}{p} + \frac{(1-\theta_c)r}{q} \geq 1$.

(ii) If $\frac{a+N}{q} = \frac{b-p+N}{p}$ *and* $c = c^0$ $(= c^1)$, *then* $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ *if* $\min\{p,q\} \leq r \leq \max\{p^*, q\}$. *Furthermore, there is a constant* $C > 0$ *such that, for every* $u \in W_0$,

$$||u||_{c,r} \leq C||\nabla u||_{b,p} \quad \text{if } p \leq r \leq p^*, \tag{6.6}$$

$$||u||_{c,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta} \text{ if } r = q \text{ or if } p \neq q \text{ and } \min\{p,q\} \leq r \leq \max\{p,q\}, \tag{6.7}$$

where $\theta := \frac{p(r-q)}{r(p-q)}$ *if* $p \neq q$ *and* $\theta = 0$ *if* $p = q = r$.

Proof. (i) Suppose $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. By (1.6), condition (ii) of Lemma 6.1 holds if and only if $\theta = \theta_c$. If $r = q$ and $c = c^0 = a$, so that $\theta_{c^0} = 0$, condition (i) of Lemma 6.1 holds with any δ. On the other hand, if $c \neq c^0$, then $\theta_c \in (0,1]$ and condition (i) of Lemma 6.1 holds with $\delta = \frac{b-p+N}{p} + \frac{1-\theta_c}{\theta_c}\frac{N}{q} - \frac{1}{\theta_c}\frac{N}{r}$. Hence, $\delta \leq \frac{b}{p}$ -as required in Lemma 6.1- if and only if $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$. Thus, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if also $\frac{\theta_c r}{p} + \frac{(1-\theta_c)r}{q} \geq 1$.

(ii) Suppose $\frac{a+N}{q} = \frac{b-p+N}{p}$ and let $c = c^0 = c^1$. Then, condition (ii) of Lemma 6.1 holds with any $\theta \in [0,1]$. Thus, it only remains to show that if $\min\{p,q\} \leq r \leq \max\{p^*, q\}$, then $\delta \leq \frac{b}{p}$ and $\theta \in [0,1]$ can be chosen such that $\frac{c}{r} = \theta\delta + (1-\theta)\frac{a}{q}$ and that $\frac{\theta r}{p} + \frac{(1-\theta)r}{q} \geq 1$. If so, all the requirements of Lemma 6.1 are satisfied, whence $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$.

Observe that $\min\{p,q\} \leq r \leq \max\{p^*, q\}$ if and only if either $p \leq r \leq p^*$ or $p \neq q$ and $\min\{p,q\} \leq r \leq \max\{p,q\}$ (possibly both). If $p \leq r \leq p^*$, we may choose $\delta = \frac{c}{r} = \frac{b}{p} + \frac{N}{p} - \frac{N}{r} - 1 \leq \frac{b}{p}$ (since $r \leq p^*$) and $\theta = 1$, so that $\frac{\theta r}{p} + \frac{(1-\theta)r}{q} = \frac{r}{p} \geq 1$. Then, (6.6) follows from (6.2).

If now $p \neq q$ and $\min\{p,q\} \leq r \leq \max\{p,q\}$, let θ be defined by $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$, i.e., $\theta = \frac{p(q-r)}{r(q-p)}$. Obviously, $\frac{\theta r}{p} + \frac{(1-\theta)r}{q} = 1$, but it must be checked that $\frac{c}{r} = \theta\delta + (1-\theta)\frac{a}{q}$ for some $\delta \leq \frac{b}{p}$. Since $c = c^0$ and $\frac{a+N}{q} = \frac{b-p+N}{p}$, a straightforward verification shows that $\frac{c}{r} = \theta\delta + (1-\theta)\frac{a}{q}$ with $\delta = \frac{b}{p} - 1$. Thus, (6.7) follows from (6.2). Of course, (6.7) remains true with $\theta = 0$ if $p = q = r$. $\square$

While Corollary 6.2 gives sufficient conditions for $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, necessary and sufficient ones for $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ are listed in Theorem 4.7, where W_{rad} is the subspace of radially symmetric functions in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Thus, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ can be inferred from the remark that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) = W_{rad} \oplus W_0$ together with the following obvious lemma:

Lemma 6.3. *Let* X *and* Y *be normed spaces and let* X_1 *and* X_2 *be two subspaces of* X *such that* $X = X_1 \oplus X_2$ *(topological direct sum). Then,* $X \hookrightarrow Y$ *if and only if* $X_i \hookrightarrow Y, i = 1, 2$.

The relation $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) = W_{rad} \oplus W_0$ reflects the equality $u = u_S + (u - u_S)$ with u_S the radial symmetrization of u, that is, $u_S(x) = f_u(|x|)$ with f_u given by (3.1). Then, $u_S \in W_{rad}$ and $||u_S||_{\{a,b\},(q,p)} \leq ||u||_{\{a,b\},(q,p)} \leq ||u||_{a,q} + ||\nabla u||_{b,p}$ by part (ii) of Lemma 3.4, which proves the continuity of $u \mapsto u_S$ ($W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ and

$\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ contain the same radially symmetric functions and the induced norms are the same). That $u - u_S \in W_0$ and $W_{rad} \cap W_0 = \{0\}$ is trivial.

The principle outlined above is simple, but it cannot always be implemented in a straightforward way, primarily because the condition (i-2) in Corollary 6.2 is far from being necessary. The case when $r < \min\{p, q\}$ (Section 5) is one, but not the only, example. In practice, this means that Corollary 6.2 alone does not always suffice to prove that $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ under optimal conditions about c. Other arguments will be needed, most notably Theorem 5.2 (but with other parameters); see the proofs of Lemma 7.2 and of Theorem 9.1.

7. Embedding theorem when $p < r \leq q$

In this section, we discuss the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $p < r \leq q$. Together with Theorem 5.2 (when $1 \leq r \leq \min\{p, q\}$), this will settle the issue when $1 \leq r \leq q$.

Theorem 7.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < r \leq q < \infty$ be given. Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) in the following cases:*
(i) a and $b-p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is in the open interval with endpoints c^0 and c^1 and $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.
(ii) a and $b-p$ are strictly on opposite sides of $-N$ (hence $\frac{a+N}{q} \neq \frac{b-p+N}{p}$), c is in the open interval with endpoints c^0 and $-N$ and $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.
(iii) $r = q$ and $c = a$.
(iv) $r \leq p^*, a \leq -N$ and $b - p < -N$ or $a \geq -N$ and $b - p > -N, c = c^1$.
(v) $\frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$ and $c = c^1$ ($= c^0$).

7.1. **Proof of parts (i) and (ii).** In this subsection, we assume $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. Let $0 \leq \bar{\theta} \leq 1$ denote the largest value of θ such that $\theta \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$, that is, since $r \leq q$ is assumed,

$$\bar{\theta} = \begin{cases} 1 \text{ if } r \leq p^*, \\ \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right)^{-1} < 1 \text{ if } p < N \text{ and } r > p^* \end{cases} \tag{7.1}$$

and call $\bar{c}$ the corresponding value of c, namely,

$$\bar{c} := \bar{\theta} c^1 + (1 - \bar{\theta}) c^0 \tag{7.2}$$

(so that $\bar{\theta} = \theta_{\bar{c}}$; see (1.4)). Since $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, the points c^0 and $\bar{c}$ coincide if and only if $\bar{\theta} = 0$, i.e., $r = q > p^*$, and then $\bar{c} = c^0 = a$ by (1.3).

Lemma 7.2. *If $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ and $\bar{c}$ is given by (7.1) and (7.2), the subspace W_0 of $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ in (6.1) is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ for every c in the interval J with endpoints $\bar{c}$ (included) and c^0 (not included, unless $r = q$).*

Proof. If $r = q$, the embedding $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for $c \in J$ follows from part (i) of Corollary 6.2 since $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q} = 0$ by definition of J and $\frac{\theta_c r}{p} + (1 - \theta_c) \geq 1$ irrespective of $\theta_c \in [0, 1]$ since $r > p$ is assumed.

From now on, $r < q$ and $c^0 \notin J$. Observe that the set $\{c \in \mathbb{R} : W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)\}$ is always an interval (in this statement, W_0 may be replaced by any normed space of measurable functions on $\mathbb{R}^N$). Thus, to prove that this interval contains J, it suffices to show that $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $c = \bar{c}$ and when $c \in J$ is arbitrarily close to c^0.

The embedding $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^{\bar{c}} dx)$ follows once again from part (i) of Corollary 6.2 since $\bar{\theta}\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$ by definition of $\bar{\theta}$ and $\frac{\bar{\theta} r}{p} + \frac{(1-\bar{\theta})r}{q} \geq 1$ by a simple calculation (obvious if $\bar{\theta} = 1$; otherwise, use $p < N$ and $q > r > p^*$).

To complete the proof, assume that $c \in J$ is close to c^0, so that $\theta_c > 0$ is small. If so, condition (i-2) of Corollary 6.2 fails when $r < q$ and this corollary cannot be used. Nonetheless, we shall prove by another argument that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ in this case, a stronger result than actually needed.

Define $\tilde{c} := \frac{(b-p)(q-r)+a(r-p)}{q-p}$ and note that, by (1.6), $\theta_{\tilde{c}} = \frac{p(q-r)}{r(q-p)} \in (0,1)$ (recall $p < r < q$), so that $\tilde{c} \neq c^0$. Both the open intervals with endpoints c^0 and $\tilde{c} \neq c^0$ or $\bar{c} \neq c^0$ consist of convex combinations of c^0 and c^1. Thus, they intersect along a nontrivial open interval having c^0 as an endpoint. As a result, it suffices to show that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c close enough to c^0 in the open interval $\widetilde{J}$ with endpoints c^0 and $\tilde{c}$.

Given any such c, set $\sigma := c - a\frac{r-p}{q-p}$ and $\gamma := \frac{q-r}{q-p} \in (0,1)$. If $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, write $|x|^c|u|^r = |x|^\sigma |u|^{p\gamma}|x|^{c-\sigma}|u|^{r-p\gamma}$ and use Hölder's inequality to get

$$\int_{\mathbb{R}^N} |x|^c |u|^r dx \leq \left(\int_{\mathbb{R}^N} |x|^{\frac{\sigma}{\gamma}} |u|^p dx\right)^\gamma \left(\int_{\mathbb{R}^N} |x|^a |u|^q dx\right)^{1-\gamma}. \tag{7.3}$$

By parts (i) and (ii) of Theorem 5.2 with c replaced by d and r replaced by p (since $p = \min\{p,q\}$), there is a nonempty open interval I with endpoint $d^0 := \frac{p(a+N)}{q} - N$ and second endpoint between d^0 and $d^1 := b - p$ (specifically, $b-p$ or $-N$), such that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^p(\mathbb{R}^N; |x|^d dx)$ when $d \in I$.

When c is moved from c^0 to $\tilde{c}$, the point $d := \frac{\sigma}{\gamma} = \frac{c(q-p)-a(r-p)}{q-r}$ moves from d^0 to $b-p$. Therefore, $d \in I$ for c in some nonempty open subinterval $\widetilde{I}$ of $\widetilde{J}$ having c^0 as an endpoint. From the above, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^p(\mathbb{R}^N; |x|^d dx)$ when $c \in \widetilde{I}$. By Corollary 2.2, this embedding is accounted for by a multiplicative inequality of the type (2.3) (with c replaced by d and r replaced by p), namely $||u||_{d,p} \leq C||\nabla u||^{\theta_d}_{b,p}||u||^{1-\theta_d}_{a,q}$ with $\theta_d := \frac{d-d^0}{d^1-d^0}$. Since $d = \frac{\sigma}{\gamma}$, the substitution into (7.3) yields, when $c \in \widetilde{I}$, the inequality $||u||_{c,r} \leq C||\nabla u||^\nu_{b,p}||u||^{1-\nu}_{a,q}$ for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, where $\nu = \frac{p\gamma\theta_d}{r} \in (0,1)$. In turn, this implies a corresponding additive (i.e., embedding) inequality. □

Proof of part (i): If $\bar{\theta} = 0$ in (7.1) (so that $r = q > p^*$), no c in the open interval with endpoints $c^0 = a$ and $c^1 = \frac{q(b-p+N)}{p} - N$ satisfies $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$ since $\theta_c > 0$ and $\theta_c \leq \bar{\theta} = 0$ are contradictory. Thus, there is nothing to prove.

If $0 < \bar{\theta} \leq 1$, so that $\bar{c} \neq c^0$, Lemma 7.2 ensures that $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the semi-open interval J with endpoints $\bar{c}$ (included) and c^0 (not included, unless $r = q$). Meanwhile, by part (i) of Theorem 4.7, $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the open interval with endpoints c^0 and c^1 (since $\breve{\theta} \leq 0$ when $r \leq q$). Thus, by Lemma

6.3, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the intersection of these two intervals. By definition of $\bar{\theta}$, this intersection is the set of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.

Proof of part (ii): Once again, it is not restrictive to assume $0 < \bar{\theta} \leq 1$. By part (ii) of Theorem 4.7, $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the open interval with endpoints c^0 and $-N$ (since $\breve{\theta} \leq 0$) and, by Lemma 7.2, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the semi-open interval with endpoints c^0 and $\bar{c}$ ($\neq c^0$ since $\bar{\theta} > 0$), including $\bar{c}$ but not c^0. Hence, by Lemma 6.3, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the intersection of these two intervals, which is the set of those c in the open interval with endpoints c^0 and $-N$ such that $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.

7.2. **Proof of parts (iii), (iv) and (v).** Since part (iii) is obvious, it only remains to prove (iv) and (v). If $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ (so that $c^1 \neq c^0$), the proof of (iv) follows from Lemma 6.3, from part (iii) of Theorem 4.7 and from part (i) of Corollary 6.2 (recall $\theta_{c^1} = 1$ and $p < r \leq p^*$). The use of part (ii) of Corollary 6.2 instead of part (i) yields (v), which in turn implies (iv) when $\frac{a+N}{q} = \frac{b-p+N}{p}$.

8. Embedding theorem when $r > q \geq 1$ and $r \geq p$

Throughout this section, we assume $r > q \geq 1$ and $r \geq p$. If also ($p < N$ and) and $r > p^*$, it follows from Theorem 2.3 and part (i) of Theorem 2.1 that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ is not continuously embedded into any $L^r(\mathbb{R}^N; |x|^c dx)$. Thus, it is not restrictive to confine attention to the case when $r \leq p^*$.

If $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, the combination $r > q$ and $q < p^*$ (i.e., $\frac{1}{q} + \frac{1}{N} - \frac{1}{p} > 0$) shows that the necessary condition for the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ given in part (i) of Theorem 2.3 is $\theta_c \geq \bar{\theta} > 0$ where

$$\bar{\theta} = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right)^{-1}. \tag{8.1}$$

This formula is the same as in (7.1), but now $\bar{\theta}$ is the *smallest* value of $\theta \in [0,1]$ such that $\theta\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$. Note that indeed $\bar{\theta} \leq 1$ because $r \leq p^*$ (and $\bar{\theta} = 1$ if and only if $r = p^*$). Equivalently, c must belong to the closed interval with endpoints $\bar{c} := \bar{\theta} c^1 + (1 - \bar{\theta}) c^0$ (as in (7.2)) and c^1.

In addition, $p \leq r < \infty$ ensures that the subspace W_0 in (6.1) is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ for c in the closed interval with endpoints $\bar{c}$ and c^1. This follows from part (i) of Corollary 6.2 since $\frac{r\theta_c}{p} + \frac{r(1-\theta_c)}{q} \geq 1$ irrespective of $\theta_c \in [0,1]$. We record this result for future reference.

Lemma 8.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$, be such that $r > q \geq 1, r \geq p$ and $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. If $\bar{c}$ is given by (8.1) and (7.2), then $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the closed interval with endpoints $\bar{c}$ and c^1.*

Lemma 8.2. *Let $a, b \in \mathbb{R}$ and $1 \leq p, r < \infty, 1 \leq q < r \leq p^*$ be such that $\frac{a+N}{q} \neq \frac{b-p+N}{p}$. If $\breve{\theta} := \left(1 - \frac{q}{r}\right)\left(\frac{q}{p'} + 1\right)^{-1}$ and $\bar{\theta}$ is given by (8.1), then $0 < \breve{\theta} \leq \bar{\theta}$.*

Proof. An explicit calculation (using $q < p^*$). □

Theorem 8.3. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be such that $1 \leq q < r$ and $r \geq p$. Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) in the following cases:*
(i) a and $b-p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is in the open interval with endpoints c^0 and c^1 and $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.
(ii) a and $b-p$ are strictly on opposite sides of $-N$ (hence $\frac{a+N}{q} \neq \frac{b-p+N}{p}$), c is in the open interval with endpoints c^0 and $-N$ and $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.
(iii) $r \leq p^*$ and either $a \leq -N$ and $b-p < -N$, or $a \geq -N$ and $b-p > -N$, $c = c^1$.
(iv) $a = -N, b = p - N, q < r \leq p^*$ and $c = c^0$ $(= c^1 = -N)$.

Proof. (i) If $r > \max\{q, p^*\}$, the condition $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$ never holds and there is nothing to prove. Accordingly, assume $r \leq p^*$, so that $\bar{\theta} \leq 1$. Since $\breve{\theta} \leq \bar{\theta}$, it follows from part (i) of Theorem 4.7 that $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for every c in the semi-open interval J with endpoints $\bar{c}$ (included) and c^1 (not included). Therefore, by Lemma 8.1 and Lemma 6.3, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for $c \in J$. By definition of $\bar{\theta}$, it is plain that J consists of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.

(ii) As in (i), it is not restrictive to assume $r \leq p^*$. Then, $\breve{\theta} \leq \bar{\theta}$ by Lemma 8.2 while $\theta_c \geq \bar{\theta}$ for every c satisfying the specified conditions. Thus, the result follows from part (ii) of Theorem 4.7, Lemma 8.1 and Lemma 6.3.

(iii) Use part (ii) of Corollary 6.2, part (iii) of Theorem 4.7 and Lemma 6.3.

(iv) Use part (ii) of Corollary 6.2, part (v) of Theorem 4.7 and Lemma 6.3. $\square$

9. Embedding theorem when $1 \leq q < r < p$

If $q < r < p$, then $r < p^*$. Thus, as in the previous section, $\bar{\theta}$ in (8.1) is the smallest $\theta \in [0, 1]$ such that $\theta \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$. Clearly, $\bar{\theta} \in (0, 1)$.

Theorem 9.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq q < r < p < \infty$ be given. Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) in the following cases:*
(i) a and $b-p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is in the open interval with endpoints c^0 and c^1 and $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.
(ii) a and $b-p$ are strictly on opposite sides of $-N$ (hence $\frac{a+N}{q} \neq \frac{b-p+N}{p}$), c is in the open interval with endpoints c^0 and $-N$ and $\theta_c \left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.
(iii) $\frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$ and $c = c^0 = c^1$.
(iv) $a = -N, b = p - N$ and $c = c^0$ $(= c^1 = -N)$.

Proof. (i) As in the proof of Lemma 7.2, let $\tilde{c} := \frac{(b-p)(q-r)+a(r-p)}{q-p}$, so that, by (1.4), $\tilde{\theta} := \theta_{\tilde{c}} = \frac{p(q-r)}{r(q-p)} \in (0, 1)$. If c is in the semi-open interval with endpoints c^0 (not included) and $\tilde{c}$ (included), then $0 < \theta_c \leq \tilde{\theta}$. A routine verification reveals that condition (i-2) of Corollary 6.2 holds but condition (i-1) holds if and only if $\theta_c \geq \bar{\theta}$ and $\tilde{\theta} > \bar{\theta}$ by another simple verification. Thus, by Corollary 6.2, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the closed interval K with endpoints $\bar{c}$ in (7.2) and $\tilde{c}$.

By part (i) of Theorem 4.7, $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the semi-open interval with endpoints $\breve{c} := \breve{\theta} c^1 + (1-\breve{\theta})c^0$ (included) and c^1 (not included) and, by Lemma 8.2, this interval contains K. Thus, by Lemma 6.3, it follows that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $c \in K$.

This is not yet the desired result, but $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{\tilde{c}} dx)$ since $\tilde{c} \in K$, so that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{\tilde{c},b\}}(\mathbb{R}^N_*)$. Now, $\frac{\tilde{c}+N}{r} \neq \frac{b-p+N}{p}$ (because $\tilde{\theta} < 1$) and $\tilde{c}$ and $b-p$ are on the same side of $-N$ (because the same thing is true of a and $b-p$). Therefore, by part (i) of Theorem 5.2 with a and q replaced by $\tilde{c}$ and r, respectively (use $r = \min\{p, r\}$), $W^{1,(r,p)}_{\{\tilde{c},b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the open interval with endpoints $\tilde{c}$ and c^1.

Altogether, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the union of K with the open interval with endpoints $\tilde{c}$ and c^1, that is, the semi-open interval with endpoints $\bar{c}$ and c^1. By definition of $\bar{\theta}$, this interval is the set of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.

(ii) If $\theta_{-N} \leq \bar{\theta}$, there is nothing to prove since no c satisfies the required conditions. Suppose then $\theta_{-N} > \bar{\theta}$. As in the proof of (i) above, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the (nontrivial) closed interval K with endpoints $\bar{c}$ and $\tilde{c}$. On the other hand, since $\breve{\theta} \leq \bar{\theta}$ by Lemma 8.2 and $\bar{\theta} < \theta_{-N}$, it follows from part (ii) of Theorem 4.7 that $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the semi-open interval $\breve{J}$ with endpoints $\breve{c} := \breve{\theta} c^1 + (1-\breve{\theta})c^0$ (included) and $-N$ (not included). Therefore, by Lemma 6.3, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $c \in K \cap \breve{J}$.

Since $\breve{\theta} \leq \bar{\theta} < \theta_{-N}$, it follows that $\bar{c} \in K \cap \breve{J}$ is an endpoint of $K \cap \breve{J}$. Since also $\bar{\theta} < \tilde{\theta}$, the second endpoint can only be $-N$ or $\tilde{c}$. If $\theta_{-N} \leq \tilde{\theta}$, then $K \cap \breve{J}$ is the semi-open interval with endpoints $\bar{c}$ (included) and $-N$ (not included). If $\theta_{-N} > \tilde{\theta}$, then $K \cap \breve{J}$ is the closed interval with endpoints $\bar{c}$ and $\tilde{c}$. Yet, once again, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when c is in the semi-open interval with endpoints $\bar{c}$ (included) and $-N$ (not included), as shown below. This proves the desired result since, by definition of $\bar{\theta}$, this interval consists of those c in the open interval with endpoints c^0 and $-N$ such that $\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$.

To complete the proof, note that, by (1.6), $\theta_{-N} > \tilde{\theta}$ implies that $\frac{\tilde{c}+N}{r}$ and $\frac{a+N}{q}$, and hence also $\tilde{c}+N$ and $a+N$, have the same (nonzero) sign, so that $\tilde{c}$ and $b-p$ are strictly on opposite sides of $-N$. As in the proof of (i) above, but now by part (ii) of Theorem 5.2 with a and q replaced by $\tilde{c}$ and r, respectively, it follows that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when c is in the union of the closed interval with endpoints $\bar{c}$ and $\tilde{c}$ with the open interval with endpoints $\tilde{c}$ and $-N$, that is, the semi-open interval with endpoints $\bar{c}$ (included) and $-N$ (not included), as claimed.

(iii) Use part (iv) of Theorem 4.7, part (ii) of Corollary 6.2 and Lemma 6.3.

(iv) The argument is the same as in the proof of part (iv) of Theorem 8.3. □

10. Generalized CKN inequalities

If $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, then $r \leq \max\{p^*, q\}$ by Theorem 2.3 and c is in the closed interval with endpoints c^0 and c^1 by part (i) of Theorem 2.1. If, in addition $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, it was shown in Corollary 2.2 that the embedding is

accounted for by the multiplicative inequality

$$||u||_{c,r} \leq C||\nabla u||_{b,p}^{\theta_c}||u||_{a,q}^{1-\theta_c}, \tag{10.1}$$

with θ_c given by (1.4). When $a, b, c > -N$ and $u \in C_0^\infty(\mathbb{R}^N)$, such inequalities coincide with some of the CKN inequalities proved in [6].

With no *a priori* limitation about a, b and c, but when $p = q = r = 2, c = \frac{a+b}{2} - 1$ -so that $\theta_c = \frac{1}{2}$- and $u \in C_0^\infty(\mathbb{R}_*^N)$, (10.1) was recently obtained, by variational methods, by Catrina and Costa [7] (see also [8]), with best constant C. This does not imply (10.1) for $u \in W_{\{a,b\}}^{1,(2,2)}(\mathbb{R}_*^N)$, or that the best constant is the same; see Subsection 11.3.

The CKN inequalities also incorporate the limiting case $\frac{a+N}{q} = \frac{b-p+N}{p}$ (when θ_c in (1.4) is not defined). It is therefore natural to ask whether the embedding $W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ can be characterized by similar multiplicative inequalities when $\frac{a+N}{q} = \frac{b-p+N}{p}$, so that $c = c^0$ $(= c^1)$ is the only possible value.

The next lemma is, roughly speaking, a "multiplicative" analog of Lemma 6.3. Recall the definition (6.1) of the subspace W_0 as well as the shorthand W_{rad} for the subspace of radially symmetric functions of $W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N)$.

Lemma 10.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be given. If there is $\theta \in [0, 1]$ such that $||u||_{c,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}$ for every $u \in W_{rad}$ and every $u \in W_0$, then $||u||_{c,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}$ for every $u \in W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N)$ after modifying C.*

Proof. Let $u \in W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N)$ be given. Then $u = u_S + (u - u_S)$, where $u_S \in W_{rad}$ and $u - u_S \in W_0$. By part (ii) of Lemma 3.4, $||u_S||_{a,q} \leq ||u||_{a,q}$ and $||\partial_\rho u_S||_{b,p} \leq ||\partial_\rho u||_{b,p} \leq ||\nabla u||_{b,p}$. Since $||\partial_\rho u_S||_{b,p} = ||\nabla u_S||_{b,p}$, the inequality $||u_S||_{c,r} \leq C||\nabla u_S||_{b,p}^{\theta}||u_S||_{a,q}^{1-\theta}$ yields $||u_S||_{c,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}$ after modifying C.

Also, $||u - u_S||_{a,q} \leq ||u||_{a,q} + ||u_S||_{a,q} \leq 2||u||_{a,q}$ and $||\nabla(u - u_S)||_{b,p} \leq ||\nabla u||_{b,p} + ||\nabla u_S||_{b,p} \leq M||\nabla u||_{b,p}$ for some $M > 0$ independent of u. Thus, $||u - u_S||_{c,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}$ after another modification of C. As a result, $||u||_{c,r} \leq ||u_S||_{c,r} + ||u - u_S||_{c,r} \leq 2C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}$. □

Theorem 10.2. *Let $a, b \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be such that $\frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$ and let $c = c^0 = c^1$.*
(i) If $p \leq r \leq p^*$, there is a constant $C > 0$ such that

$$||u||_{c^1,r} \leq C||\nabla u||_{b,p}, \qquad \forall u \in W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N). \tag{10.2}$$

(ii) If $r = p = q$ or if $p \neq q$ and $\min\{p, q\} \leq r \leq \max\{p, q\}$, there is a constant $C > 0$ such that

$$||u||_{c^1,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}, \qquad \forall u \in W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N), \tag{10.3}$$

where $\theta = 0$ if $r = p = q$ and $\theta = \frac{p(r-q)}{r(p-q)}$ if $p \neq q$.

Proof. (i) Use Lemma 10.1 together with the inequality (4.6) in part (iii) of Theorem 4.7 and the inequality (6.6) in part (ii) of Corollary 6.2.

(ii) Use Lemma 10.1 together with the inequality (4.7) in part (iv) of Theorem 4.7 and the inequality (6.7) in part (ii) of Corollary 6.2. □

Remark 10.1. *If $\frac{a+N}{q} = \frac{b-p+N}{p} \neq 0$ and $p \leq r \leq \min\{p^*, \max\{p,q\}\}$, (10.2) and (10.3) show that $||u||_{c^1,r} \leq C||\nabla u||_{b,p}^{\theta}||u||_{a,q}^{1-\theta}$ for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ with $\theta = 1$ and with $\theta = \underline{\theta}$, where $\underline{\theta} = \frac{p(r-q)}{r(p-q)}$ if $p \neq q$and $\underline{\theta} = 0$ if $p = r = q$. Hence, the inequality holds with $\theta \in [\underline{\theta}, 1]$ and so θ is not unique if $r > p \neq q$ or if $r = p = q$. This is actually trivial if $r = q \geq p$ (because (10.3) is trivial), but not in the other cases: $p < r < q \leq p^*$ or $p < N$ and $p < r \leq p^* < q$.*

Clearly, (10.2) is an N-dimensional weighted Hardy-type inequality, apparently new when $q \neq p$. It is proved in [22, p. 309] when $q = p$, so that $a = b - p \neq -N$. When $u \in C_0^\infty(\mathbb{R}^N_*)$, it was obtained earlier by Gatto, Gutiérrez and Wheeden [9], who showed that $p \leq r \leq p^*$ is already necessary in that setting. A number of special cases of (10.2) for various classes of smooth functions with compact support can be found in both the older and the recent literature ([10], [13], [28], among others). The inequality (10.3), meaningless when $q = p$, seems to be known only if $a, b, c > -N$ and $u \in C_0^\infty(\mathbb{R}^N)$, when it is one of the CKN inequalities.

By Corollary 2.2, the inequality (sharper than (10.1))

$$||u||_{c,r} \leq C||\partial_\rho u||_{b,p}^{\theta_c}||u||_{a,q}^{1-\theta_c}, \qquad \forall u \in \widetilde{W}^{(q,p)}_{\{a,b\}},$$

holds if $\frac{a+N}{q} \neq \frac{b-p+N}{p}$, c is in the closed interval with endpoints c^0 and c^1 and $\widetilde{W}^{(q,p)}_{\{a,b\}} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$. Necessary and sufficient conditions for this embedding were given in Theorem 5.2 when $r \leq \min\{p, q\}$, where it is also shown that $\widetilde{W}^{(q,p)}_{\{a,b\}} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if $\frac{a+N}{q} = \frac{b-p+N}{p} \neq 0, r = p$ $(\leq q)$ and $c = c^0 = c^1$. If so, it follows from part (iii) of Theorem 4.7 and from Lemma 5.1 that

$$||u||_{b-p,p} \leq C||\partial_\rho u||_{b,p}, \qquad \forall u \in \widetilde{W}^{(q,p)}_{\{a,b\}}.$$

The only case when the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ is true but *not* equivalent to a multiplicative inequality arises in part (vi) of Theorem 1.1 when $N \geq 2$ (if u is radially symmetric, or $N = 1$, see (4.8):

Theorem 10.3. *If $q < r \leq p^*$, then $W^{1,(q,p)}_{\{-N,p-N\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{-N} dx)$ but when $N \geq 2$, the inequality $||u||_{-N,r} \leq C||\nabla u||_{p-N,p}^{\theta}||u||_{-N,q}^{1-\theta}$ fails to hold for every $C > 0$ and every $\theta \in [0, 1]$.*

Proof. The embedding is part (vi) of Theorem 1.1. Also, the inequality can only hold if $\theta = \breve{\theta} := \left(1 - \frac{q}{r}\right)\left(\frac{q}{p'} + 1\right)^{-1}$. This follows by choosing $u(x) = g(\ln|x|)$ with $g \in C_0^\infty(\mathbb{R})$ and by reversing the steps of the proof of part (v) of Theorem 4.7 (by [6], (4.10) cannot hold with $\theta \neq \breve{\theta}$ when $g \in C_0^\infty(\mathbb{R})$ is arbitrary).

Next, if $||u||_{-N,r} \leq C||\nabla u||_{p-N,p}^{\theta}||u||_{-N,q}^{1-\theta}$, the method of proof of Theorem 2.3 with $a = c = -N$ and $b = p - N$ shows that $\theta\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}$. Upon substituting the only possible value $\theta = \breve{\theta}$, a short calculation yields $q(N-1) \geq r(N-1)$. If $N \geq 2$, this implies $q \geq r$, which contradicts $q < r \leq p^*$.

Consistent with Theorem 10.3 and its proof, it is easily verified that when $a = b - p = c = -N, \theta = \breve{\theta}$ and $N \geq 2$, Lemma 10.1 is not applicable because condition (i) of Lemma 6.1 fails, so that (6.2) cannot be used. $\square$

11. Examples

11.1. **Embedding of unweighted spaces into $L^r(\mathbb{R}^N;|x|^c dx)$.** We spell out the special case of Theorem 1.1 when $a=b=0$. It is noteworthy that $W^{1,(q,p)}(\mathbb{R}^N_*) = W^{1,(q,p)}(\mathbb{R}^N) = \{u \in L^q(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\}$ if $N \geq 2$, with the same norm (see Remark 11.1 later). At any rate, if $a=b=0$, then θ_c in (1.4) is defined if and only if $\frac{1}{p}-\frac{1}{N}-\frac{1}{q} \neq 0$, i.e., $q \neq p^*$ and then $\theta_c = \left(\frac{c+N}{rN}-\frac{1}{q}\right)\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right)^{-1}$. Therefore, the condition $\theta_c\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \leq \frac{1}{r}-\frac{1}{q}$ in parts (i) and (ii) of Theorem 1.1 is just $c \leq 0$. It follows that $W^{1,(q,p)}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N;|x|^c dx)$ if and only if $r \leq \max\{p^*,q\}$ and one of the following conditions holds:

(i) $p \leq N, q \neq p^*$ and $c \leq 0$ is in the open interval with endpoints $\frac{rN}{q}-N$ and $r\left(\frac{N-p}{p}\right)-N$ (a nonempty set if $r<\max\{p^*,q\}$).

(ii) $p>N$ and either $r \leq q$ and $-N<c<\frac{rN}{q}-N$ (≤ 0) or $r>q$ and $-N<c\leq 0$.

(iii) $r=q$ and $c=0$.

(iv) $p<N, p\leq r\leq p^*$ and $c=r\left(\frac{N-p}{p}\right)-N$ (≤ 0 since $r \leq p^*$).

Since $\frac{N}{q}=\frac{N}{p}-1$ implies $p<N$ and $q>p$, part (v) of Theorem 1.1 coincides with (iv) above. Part (vi) of Theorem 1.1 is not applicable.

If $r \leq \min\{p,q\}$, the conditions (i)-(iv) are necessary and sufficient for $\widetilde{W}^{1,(q,p)} \hookrightarrow L^r(\mathbb{R}^N;|x|^c dx)$, where $\widetilde{W}^{1,(q,p)} := \widetilde{W}^{1,(q,p)}_{\{0,0\}}$ is unweighted (Theorem 5.2) an they take the simpler form (i) $p\leq N, q\neq p^*$ and c is in the open interval with endpoints $\frac{rN}{q}-N$ and $r\left(\frac{N-p}{p}\right)-N$ (hence $c<0$). (ii) $p>N$ and $-N<c<\frac{rN}{q}-N$ (≤ 0). (iii) $q\leq p, r=q$ and $c=0$. (iv) $p\leq q, p<N, r=p$ and $c=-p$.

When $c=0$, the conditions become (i) $p<N$ and r is in the closed interval with endpoints p^* and q or (ii) $p\geq N$ and $r\geq q$. This is of course well-known, especially when $p=q$.

Remark 11.1. *That $W^{1,(q,p)}(\mathbb{R}^N_*) = W^{1,(q,p)}(\mathbb{R}^N)$ with the same norm if $N>1$ can be seen as follows: First, it suffices to show that if $u \in W^{1,(q,p)}(\mathbb{R}^N_*)$ has bounded support, then $u \in W^{1,(q,p)}(\mathbb{R}^N)$ with the same norm. Now, if $u \in W^{1,(q,p)}(\mathbb{R}^N_*)$ has bounded support, then $u \in W^{1,\min\{p,q\}}(\mathbb{R}^N_*) = W^{1,\min\{p,q\}}(\mathbb{R}^N)$, for example by* [11, p. 52]*. Thus, as a distribution on $\mathbb{R}^N, \nabla u$ is a function, so that its restriction to $\mathbb{R}^N_*$ coincides with ∇u as a distribution on $\mathbb{R}^N_*$. Since the latter is in $(L^q(\mathbb{R}^N))^N$, the same thing is true of the former, which proves the claim.*

11.2. **Embedding of weighted spaces into $L^r(\mathbb{R}^N)$.** The necessary and sufficient conditions for $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N)$ are given by Theorem 1.1 with $c=0$. If so, $\theta_0 = \left(\frac{N}{r}-\frac{a+N}{q}\right)\left(\frac{b-p+N}{p}-\frac{a+N}{q}\right)^{-1}$ in (1.4) when $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ and these conditions become (after some work) $r \leq \max\{p^*,q\}$ and

(i)-(ii) Either $-N \leq a < N\left(\frac{q}{r}-1\right), b>p+N\left(\frac{p}{r}-1\right)$and $a\left(\frac{p}{r}-1+\frac{p}{N}\right) \leq b\left(\frac{q}{r}-1\right)$, or $a>N\left(\frac{q}{r}-1\right), b<p+N\left(\frac{p}{r}-1\right)$ and $a\left(\frac{p}{r}-1+\frac{p}{N}\right) \geq b\left(\frac{q}{r}-1\right)$.

(iii) $r=q$ and $a=0$.

(iv) $p\leq r\leq p^*, b=p+N\left(\frac{p}{r}-1\right)$ $(\leq p)$ and $a\geq -N$.

(v) $r\geq \min\{p,q\}, a=N\left(\frac{q}{r}-1\right)$ and $b=p+N\left(\frac{p}{r}-1\right)$.

In (i)-(ii) above, the condition $\theta_0\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \leq \frac{1}{r}-\frac{1}{q}$ is accounted for by $a\left(\frac{p}{r}-1+\frac{p}{N}\right) \leq b\left(\frac{q}{r}-1\right)$ or its reverse, as the case may be. By Remark 1.1, this condition holds if $r \leq \min\{p^*, q\}$, which of course is corroborated by a direct verification.

11.3. **Embedding when** $p = q$**.** If $p = q$, then $r \leq \max\{p^*, q\}$ is simply $r \leq p^*$ and $\frac{a+N}{q} \neq \frac{b-p+N}{p}$ if and only if $a \neq b-p$. The condition $\theta_c\left(\frac{1}{p}-\frac{1}{N}-\frac{1}{q}\right) \leq \frac{1}{r}-\frac{1}{q}$ in parts (i) and (ii) of Theorem 1.1 becomes $\theta_c \geq \frac{N}{p}-\frac{N}{r}$, which is not a restriction when $r \leq p$. Also, part (v) is now a special case of part (iv).

If, in addition, $p = q = r$, Theorem 5.2 is applicable to the larger space $\widetilde{W}^{1,(p,p)}_{\{a,b\}}$. Furthermore, $c^0 = a$ and $c^1 = b-p$ and so $\widetilde{W}^{1,(p,p)}_{\{a,b\}} \hookrightarrow L^p(\mathbb{R}^N; |x|^c dx)$ if and only if either (i) a and $b-p$ are on the same side of $-N$, not both equal to $-N$, and c is in the closed interval with endpoints a and $b-p$ or (ii) a and $b-p$ are strictly on opposite sides of $-N$ and c is in the semi-open interval with endpoints a (included) and $-N$ (not included). These are also necessary and sufficient conditions for $W^{1,(p,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^p(\mathbb{R}^N; |x|^c dx)$.

When $p = q = r = 2$ and $c = \frac{a+b}{2}-1$, it follows from [7] that $C^\infty_0(\mathbb{R}^N_*) \hookrightarrow L^2(\mathbb{R}^N; |x|^c dx)$, unless $a = b-2 = -N$. If (for example) $a < -N$ and $b > -a+2-2N$, then $b-2 > -N$, so that a and $b-2$ are on opposite sides of $-N$ but, since $c = \frac{a+b}{2}-1 > -N$, condition (ii) above does not hold and so $W^{1,(2,2)}_{\{a,b\}}(\mathbb{R}^N_*)$ is not continuously embedded into $L^2(\mathbb{R}^N; |x|^c dx)$. This shows that $C^\infty_0(\mathbb{R}^N_*)$ is not dense in $W^{1,(2,2)}_{\{a,b\}}(\mathbb{R}^N_*)$. Accordingly, in general, embedding (or other) inequalities for $W^{1,(p,q)}_{\{a,b\}}(\mathbb{R}^N_*)$ cannot be proved by confining attention to $C^\infty_0(\mathbb{R}^N_*)$.

11.4. **A generalization.** Let $B \subset \mathbb{R}^N$ be an open ball centered at the origin. If the space $\{u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) : \operatorname{Supp} u \subset \overline{B}\}$ is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$, it is also continuously embedded into $L^r(\mathbb{R}^N; |x|^d dx)$ when $d \geq c$. Likewise, if $\{u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) : \operatorname{Supp} u \subset \mathbb{R}^N \backslash B\}$ is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$, it is also continuously embedded into $L^r(\mathbb{R}^N; |x|^d dx)$ when $d \leq c$.

With this remark and a cut-off argument, Theorem 1.1 can be extended to more general weighted spaces. Let $x_1, ..., x_k \in \mathbb{R}^N$ be distinct points and let $a_1, ..., a_k, a_\infty, b_1, ..., b_k, b_\infty \in \mathbb{R}$ and $1 \leq r \leq p, q < \infty$ be given. For $a, b \in \mathbb{R}$, call $J(a,b) := \{c \in \mathbb{R} : W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)\}$ the interval of admissible c characterized in Theorem 1.1, with endpoints $c_-(a,b) \leq c_+(a,b)$ and let $c_1, ..., c_k, c_\infty$ be such that $c_i > c_-(a_i, b_i), 1 \leq i \leq k$ and $c_\infty < c_+(a_\infty, b_\infty)$ (the endpoints may be included if they are in the admissible interval). If w_a, w_b and[6] w_c are positive weights on $\mathbb{R}^N \backslash \{x_1, ..., x_k\}$ such that $w_a(x) = |x-x_i|^{a_i}, w_b(x) = |x-x_i|^{b_i}, w_c(x) = |x-x_i|^{c_i}$ for x near $x_i, i = 1, ..., k$ and $w_a(x) = |x|^{a_\infty}, w_b(x) = |x|^{b_\infty}, w_c(x) = |x|^{c_\infty}$ for large $|x|$, then the space $W^{1,(q,p)}_{\{w_a,w_b\}}(\mathbb{R}^N \backslash \{x_1, ..., x_k\}) := \{u \in L^1_{loc}(\mathbb{R}^N \backslash \{x_1, ..., x_k\}) : u \in L^q(\mathbb{R}^N; w_a(x)dx), \nabla u \in (L^q(\mathbb{R}^N; w_b(x)dx))^N\}$ is continuously embedded into $L^r(\mathbb{R}^N; w_c(x)dx)$.

A somewhat heuristic but compelling reason why such conditions should be optimal is simple: As pointed out above, the membership to $L^r(\mathbb{R}^N; |x|^c dx)$ of

[6] Here, a, b and c are just indices.

functions with support in a closed ball $\overline{B}$ about the origin is unaffected by increasing c. Thus, the value of the upper end $c_+(a,b)$ can only be dictated by the behavior of functions with support bounded away from 0. The optimality of the lower end $c_-(a,b)$ is justified by a similar argument. However, this rationale is meaningless when $J(a,b) = \emptyset$. If so, the simplest way around the difficulty is to rely on the related fact that for functions with support in $\overline{B}$, membership to $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ is unaffected by increasing a or b, so that doing so until $J(a,b)$ becomes nonempty can be used to define $c_-(a,b)$. Likewise, a or b can be decreased to define $c_+(a,b)$. This may or may not produce the best possible conditions. Due to space limitations, a more detailed investigation of the optimality issue by more sophisticated procedures (elaboration on Remark 4.3) will not be attempted here.

Naturally, the weights need only to "look like" (not coincide with) power weights in the vicinity of the points x_i (or infinity). This remark clarifies two things. First, w_a, w_b and w_c need actually not have power-like singularities at the same points: This case is reduced to the previous one by adding points as needed and setting the corresponding a_i, b_i or c_i equal to 0. Next, the cut-off argument is technically simplified, and nothing is changed, if it is assumed that $w_a(x) = |x-x_1|^{a_\infty}, w_b(x) = |x-x_1|^{b_\infty}, w_c(x) = |x-x_1|^{c_\infty}$ for large $|x|$ (otherwise, the origin plays a technical role even when it is not one of the points x_i). Theorem 1.1 is recovered when $k=1, x_1 = 0$ and $a_1 = a_\infty, b_1 = b_\infty, c_1 = c_\infty$.

If only $k=1$ and $x_1 = 0$, Theorem 4.7 too can be generalized to obtain the embedding of the subspace W_{rad} of radially symmetric functions in $W^{1,(q,p)}_{\{w_a,w_b\}}(\mathbb{R}^N_*)$ into $L^r(\mathbb{R}^N; w_c(x)dx)$ under the conditions $c_1 > c^{rad}_-(a_1,b_1)$ and $c_\infty < c^{rad}_+(a_\infty,b_\infty)$, where $c^{rad}_\pm(a,b)$ denote the endpoints of the admissible interval in Theorem 4.7. Once again, $c^{rad}_-(a_1,b_1)$ and $c^{rad}_+(a_\infty,b_\infty)$ may be included if they are in the admissible interval and they can also be defined when the admissible interval is empty.

When $1 < p = q < N$ and $w_b = 1$ (so that $b_1 = b_\infty = 0$), the embedding into $L^r(\mathbb{R}^N; w_c(x)dx)$ of the closure C_{rad} of the space of radially symmetric functions in $C^\infty_0(\mathbb{R}^N) \cap L^p(\mathbb{R}^N; w_a(x)dx)$ equipped with the $W^{1,(p,p)}_{\{w_a,1\}}(\mathbb{R}^N_*)$ norm, has recently been investigated by Su *et al.* [28, Theorems 1 and 2]. They assume that $a_1, c_1, a_\infty, c_\infty$ are given and find the admissible values of r under the implicit assumption $r \geq p$. The reformulation in terms of lower (upper) bounds about c_1 (c_∞) given a_1, a_∞ and r is conceptually trivial, but quite messy and technical in practice. Accordingly, we shall not elaborate beyond the remark that, because C_{rad} is usually *smaller* than W_{rad}, the embedding may be true under conditions more general than $c_1 \geq c^{rad}_-(a_1,1)$ and $c_\infty \leq c^{rad}_+(a_\infty,1)$. On the other hand, the case $0 < r < p$ and all others ($p = 1, p \geq N, q \neq p, b_1 \neq 0, b_\infty \neq 0$) can be handled by the method outlined above.

References

[1] Bakry, D., Coulhon, T., Ledoux, M. and Saloff-Coste, L., Sobolev inequalities in disguise, *Indiana Univ. Math. J.* **44** (1995) 1033-1074.

[2] Besov, O. V., Il'in, V. P. and Nikol'skii, S. M., *Integral representation of functions and imbedding theorems,* Nauka Moscow 1975, English translation, Vol. I, Wiley, New York 1978.

[3] Besov, O. V., Il'in, V. P. and Nikol'skii, S. M., *Integral representation of functions and imbedding theorems,* Nauka Moscow 1975, English translation, Vol. II, Wiley, New York 1979.

[4] Benyamini, Y. and Lindenstrauss, J., *Geometric nonlinear functional analysis, Vol. 1,* Colloquium Publications **48**, American Mathematical Society, Providence, 2000.

[5] Bradley, S. C., Hardy inequalities with mixed norms, *Canad. Math. Bull.* **21** (1978) 405-408.
[6] Caffarelli, L., Kohn, R. and Nirenberg, L., First order interpolation inequalities with weights, *Compos. Math.* **53** (1984) 259-275.
[7] Catrina, F. and Costa, D. G., Sharp weighted-norm inequalities for functions with compact support in $\mathbb{R}^N\backslash\{0\}$, *J. Differential Equations* **246** (2009) 164-182.
[8] Costa, D. G., Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities, *J. Math. Anal. Appl.* **337** (2008) 311-317.
[9] Gatto, A. E., Gutiérrez, C. E. and Wheeden, R. L., Fractional integrals on weighted H^p spaces, *Trans. Amer. Math. Soc.* **289** (1985) 575-589.
[10] Glaser, V., Martin, A., Grosse H. and Thirring, W., A family of optimal conditions for the absence of bound states in a potential, *Studies in Mathematical Physics,* Princeton University Press (1976)169-194.
[11] Heinonen, J., Kilpeläinen, T. and Martio, O., *Nonlinear potential theory of degenerate elliptic equations,* Oxford Univ. Press, Oxford 1993.
[12] Il'in, V. P., Some integral inequalities and their applications in the theory of differentiable functions of several variables (Russian), *Mat. Sb. (N.S.)* **54** (1961) 331-380.
[13] Lieb, E., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.* **118** (1983) 349-374.
[14] Marcus, M. and Mizel, V. J., Absolute continuity on tracks and mappings of Sobolev spaces, *Arch. Rational Mech. Anal.* **45** (1972) 294-320.
[15] Maz'ja, V. G., Classes of domains and imbedding theorems for function spaces, *Soviet Math. Dokl.* **1** (1960) 882-885.
[16] Maz'ja, V. G., On certain integral inequalities for functions of many variables, *J. Math. Sciences* **1** (1973) 205-234.
[17] Maz'ja, V. G., *Sobolev spaces*, Springer-Verlag, Berlin, 1985.
[18] Maz'ya, V., *Sobolev spaces with applications to elliptic partial differential equations,* Grund. Math. Wiss. Vol. 342, Springer, Heidelberg 2011.
[19] Muckenhoupt, B., Hardy's inequality with weights, *Studia Math.* **44** (1972) 31-38.
[20] Nakai, E., Tomita, N. and Yabuta, K., Density of the set of all infinitely differentiable functions with compact support in weighted Sobolev spaces, *Sci. Math. Jpn.* **60** (2004) 121-127.
[21] Opic, B. and Gurka, P., N-dimensional Hardy inequality and imbedding theorems for weighted Sobolev spaces on unbounded domains, in: *Function spaces, differential operators and nonlinear analysis* 108-124, Pitman Research Notes in Mathematics **211**, Longman, Harlow 1989.
[22] Opic, B. and Kufner, A., *Hardy-type inequalities,* Pitman Research Notes in Mathematics **219**, Longman, Harlow 1990.
[23] Rákosník, J. Some remarks to anisotropic Sobolev spaces. I. *Beiträge Anal.* **13** (1979) 55-68.
[24] Rolewicz, S., *Metric linear spaces,* 2^{nd} ed., Mathematics and its Applications (East European Series) **20**, D. Reidel Publishing Co., Dordrecht, 1985.
[25] Serrin, J. and Varberg, D. E., A general chain rule for derivatives and the change of variables formula for the Lebesgue integral, *Amer. Math. Monthly* **76** (1969) 514-520.
[26] Sinnamon, G., Weighted Hardy and Opial-type inequalities, *J. Math. Anal. Appl.* **160** (1991) 434-445.
[27] Sinnamon, G. and Stepanov, V. D., The weighted Hardy inequality: New proofs and the case $p = 1$, *J. London Math. Soc.*, **54** (1996) 89-101.
[28] Su, J., Wang, Z.-Q. and Willem, M., Weighted Sobolev embedding with unbounded and decaying radial potentials, *J. Differential Equations* **238** (2007) 201-219.
[29] de la Vallée Poussin, Sur l'intégrale de Lebesgue, *Trans. Amer. Math. Soc.* **16** (1915) 435-501.
[30] Zhikov, V. V., On weighted Sobolev spaces, *Sb. Math.* **189** (1998) 1139-1170.
[31] Ziemer, W. P., *Weakly differentiable functions,* Springer-Verlag, Berlin, 1989.

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: rabier@imap.pitt.edu